Stationary Rational Bubbles in Non-Linear Business Cycle Models of Closed and Open Economies

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This paper shows that multiple stationary equilibria can exist in standard *non-linear* DSGE models, even when the *linearized* versions of those models have a unique solution. Thus, the non-linear models can exhibit stationary fluctuations, even if there are no shocks to productivity, preferences or other 'fundamentals'. In the equilibria considered here, the economy may temporarily diverge from the steady state, before abruptly reverting towards the steady state. In contrast to rational bubbles in linear models (Blanchard (1979)), the rational bubbles in non-linear models considered here are stationary--their path does not explode to ±∞. Numerical simulations suggest that non-linear DSGE models driven solely by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts. Applications to both closed and open economies are analyzed. A key finding for a multi- country model is that, with integrated financial markets, investment bubbles have to be perfectly correlated across countries. Global bubbles may help to explain the synchronization of international business cycles. Country-specific bubbles can only arise when there are impediments to international capital flows.

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1. Introduction

Linearized dynamic stochastic general equilibrium (DSGE) models with a unique non-explosive solution are the workhorses of modern quantitative macroeconomics (e.g., King and Rebelo, 1999; Kollmann et al., 2011a,b).This paper shows that multiple stationary equilibria may exist in standard non-linear DSGE models, even when the linearized versions of those models have a unique solution. Thus, the non-linear models can exhibit stationary fluctuations of endogenous variables, even if there are no shocks to productivity, preferences or other exogenous 'fundamentals'. The Blanchard and Kahn (1980) conditions for the uniqueness of stable solutions to linear rational expectations models are, hence, irrelevant for non-linear models. In the equilibria considered here, the economy may temporarily diverge from the steady state, but with some (exogenous) probability the economy reverts towards the steady state later. These boom-bust cycles are consistent with rational expectations. Importantly, the 'rational bubbles' studied here are stationary.

The multiple equilibria identified here have similarities and important differences, compared to the rational bubbles in linearized models analyzed by Blanchard (1979). Like Blanchard (1979), the study here focuses on models whose linearized versions have a unique non-explosive equilibrium. Like the Blanchard bubbles, the rational bubbles in non-linear models discussed here imply that endogenous variables can diverge from the steady state, before abruptly reverting towards the steady state. The key difference is that the bubbles in non-linear models considered here are stationary, while Blanchard's bubbles in linearized models exhibit explosive *expected* trajectories that tend to $\pm\infty$. This feature greatly limits the appeal of the Blanchard bubbles for DSGE models. In a standard DSGE model with decreasing returns to capital and capital depreciation, an explosive trajectory of the capital stock and output is infeasible. A linear model approximation does not take this into account. The accuracy of linear approximations deteriorates sharply when the state variables deviate substantially from the point of approximation—in particular, non-negativity constraints on endogenous variables and other technological feasibility restrictions may be violated. A linear model approximation is, thus, not suitable for studying rational bubbles.

By contrast, the non-linear model analysis here takes non-negativity constraints and decreasing returns into account. Decreasing returns and risk aversion generate stabilizing forces that prevent explosive trajectories. While rational bubbles in linearized models can be positive or negative, I find that rational bubbles in standard non-linear models are generally one-sided; e.g., they tend to predict over-accumulation of capital, but not under-accumulation. I show that rational bubbles in non-linear models can induce fluctuations that remain close to deterministic steady state most of the time; the unconditional mean of endogenous variables can thus be close to the deterministic steady state.

This paper analyzes bubbles in both closed and open economies. Numerical simulations suggest that non-linear DSGE models driven solely by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts. A key finding for a two-country model is that, with integrated financial markets, investment bubbles have to be perfectly correlated across countries. Global bubbles may help to explain the synchronization of international business cycles. Country-specific bubbles can only arise when there are impediments to international capital flows.

The standard DSGE models discussed in this paper are usually presented as structures with an optimizing infinitely-lived representative household. The set of optimality conditions of that household's decision problem includes a transversality condition (TVC) that stipulates that the value of capital has to be zero, at infinity. The TVC (in conjunction with Euler equations and static efficiency conditions) implies a unique equilibrium, in standard DSGE models. When TVCs do not hold, the economy is 'dynamically inefficient' (e.g., Abel et al. (1989)).

I do not impose the TVC in this paper. My goal is to show that stationary rational bubbles can exist in standard non-linear DSGE models. Note that explosive bubbles in linear models (Blanchard (1979)) likewise violate the TVC.

A possible justification for disregarding the TVC is that there is no TVC because agents are finitely-lived. I show that there exists an overlapping generations (OLG) structure with finitely-lived households that delivers the same Euler equations and static efficiency conditions as the standard DSGE models discussed here. However, the TVC does not hold in that OLG structure. The OLG structure proposed here provides thus a motivation for exploring bubble equilibria in standard DSGE models. The key features of this OLG structure are: (i) there is complete risk sharing among generations that are alive at the same dates; (ii) newborn agents receive a wealth endowment such that the consumption of newborns represents a time-invariant share of aggregate consumption (under log utility, this requires that the wealth endowments of newborns is a time-invariant fraction of aggregate wealth). The linearized version of the OLG structure presented here has a unique non-explosive solution, but the non-linear model has multiple stationary bubble equilibria.

Another motivation for disregarding the TVC is that detecting TVC violations can be very difficult in non-linear stochastic economies, for which no closed form solution exists. TVC violations may be caused by low probability events in a distant future. Households may thus lack the cognitive/computing power to detect deviations from TVC (see discussion in Blanchard and Watson (1982), Lansing (2010) and Ascari et al. (2019)).

A large literature has studied linearized DSGE models with stationary sunspot equilibria (i.e. multiple stationary equilibria). These equilibria exist (in linearized models) if the number of eigenvalues (of the linearized state-space form) outside the unit circle is less than the number of non-predetermined variables (Blanchard and Kahn (1980), Prop. 3).[1] By contrast, the paper here focuses on models for which the number of eigenvalues equals the number of non-predetermined variables, so that the linearized structure has a unique non-explosive solution (Blanchard and Kahn (1980), Prop. 1). Linearized models may exhibit stationary sunspot equilibria if increasing returns and/or externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999)), financial frictions (e.g., Martin and Ventura (2018)) or certain OLG population structures (e.g., Woodford (1986), Galí (2018)) are assumed. The specific assumptions and calibrations that deliver sunspot equilibria in linearized models can be debatable.[2] By contrast, the paper here shows that very standard DSGE models (without the features that were just mentioned) whose *linearized* versions have a *unique* stationary solution can have multiple equilibria, if non-linear effects are considered.

The role of non-linearities for multiple equilibria is also studied by Holden (2016a,b) who shows that multiple equilibria can emerge when occasionally binding constraints (such as borrowing constraints or non-negativity constraints) are integrated into an otherwise linear model (where that linear model has a unique stable solution when the occasionally binding constraints are ignored). By contrast, the analysis here considers *fully* non-linear models; the solutions considered here are globally accurate. The multiple equilibria described here have a 'bubbly' dynamics that differs from the dynamics highlighted by Holden (2016a,b).

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[1] See Taylor (1977) for an early example of a model with sunspots, due to the presence of 'too many' stable roots.

[2] E.g., increasing returns/externalities need to be sufficiently strong; in OLG models the steady state interest rate has to be smaller than the trend growth rate (r<g) etc. Note that, in the novel OLG structure developed in the paper here r>g holds. Linearized versions of the OLG structure here have a unique equilibrium (as discussed above).

The bubble equilibria discussed in this paper imply that the distribution of endogenous variables is heteroscedastic: the condition variance of future forecast errors of endogenous variables is greater, the longer a boom driven by self-fulfilling expectations has lasted. In this sense, the paper here is related to Bacchetta et al. (2012) who study a stylized asset pricing model with two-period lived agents in which stationary stock price bubbles can arise if the sunspot shock is heteroscedastic. The work here highlights the importance of heteroscedasticity, for generating stationary bubbles, in non-linear DSGE business cycle models.

The next Section discusses stationary rational bubbles in a one-sector version of the Long and Plosser (1983) model, i.e. in a closed-economy RBC model with log utility, a Cobb-Douglas production function and full capital depreciation per period. Closed form solutions with bubbles can be derived for that model. Section 3 considers a more realistic non-linear closed economy RBC model with incomplete capital depreciation. Sections 4 and 5 study stationary rational bubbles in non-linear two-country RBC models.

2. Rational bubbles in the Long-Plosser RBC model

Following Long and Plosser (1986), this Section considers a closed economy inhabited by a household with time-separable life-time utility. The period utility function is logarithmic: $u(C_t)=\ln(C_t)$, where C_t is consumption in period *t*. The production function is Cobb-Douglas:

$$Y_t=\theta_t K_t^{\alpha}\,,\ 0<\alpha<1\,, \tag{1}$$

where $Y_t, K_t, \theta_t>0$ are output, capital and exogenous total factor productivity (TFP). For simplicity, I assume that labor hours are constant and normalized at 1 (the next Sections allow for variable hours).[3] The resource constraint is

$$C_t+I_t=Y_t, \tag{2}$$

where I_t is gross investment. The capital depreciation rate is 100%, so that gross investment equals next period's capital stock: $I_t=K_{t+1}$. The household's Euler equation is

$$E_t\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1}=1\,, \tag{3}$$

[3] With endogenous hours, and a period utility function that is additively separable in consumption and hours, hours are constant in the no-bubbles solution of the Long-Plosser model, while hours fluctuate in bubble equilibria.

where $0<\beta<1$ is the household's subjective discount factor. Substitution of the resource constraint into the Euler equation gives an expectational difference equation in the investment/output ratio $Z_t \equiv K_{t+1}/Y_t$:

$$E_t H(Z_{t+1}, Z_t) \equiv E_t \alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1. \tag{4}$$

$Z_t = \alpha\beta \;\forall t$ solves (4). This corresponds to the textbook solution of the Long-Plosser model (see, e.g., Blanchard and Fischer (1989)). Under this solution, consumption and investment are constant shares of output: $C_t = (1-\alpha\beta)Y_t,\; K_{t+1} = \alpha\beta Y_t \quad \forall t$.

2.1. Bubbles in the <u>linearized</u> Long-Plosser model

Linearization of (4) around $Z = \alpha\beta$ gives:

$$E_t z_{t+1} = \lambda z_t, \text{ with } z_t \equiv Z_t - Z \text{ and } \lambda \equiv 1/(\alpha\beta) > 1. \tag{5}$$

λ, the eigenvalue of (5) exceeds unity. The model has one non-predetermined variable (z_t). Thus, the linearized model has a unique non-explosive solution (Blanchard and Kahn (1980), Proposition 1). This solution is given by $z_t = 0$, i.e. $Z_t = \alpha\beta \;\forall t$.

Blanchard (1979) pointed out that a linear expectational difference equation of form (5) is also solved by a process $\{z_t\}$ such that

$$z_{t+1} = [\lambda/(1-\pi)] \cdot z_t \text{ with probability } 1-\pi \text{ and } z_{t+1} = 0 \text{ with probability } \pi \quad (0<\pi<1). \tag{6}$$

If there is a **bubble** at date t, i.e. $z_t \neq 0$, then next period the bubble grows with probability $1-\pi$; the bubble bursts with probability π. The larger the bubble, the greater the magnitude of the subsequent 'correction'. The bubble process (6) implies that after a bubble has burst, a new bubble never arises again (the bubble is 'self-ending'). As noted by Blanchard (1979), recurrent bubbles obtain if a bursting bubble reverts to a value $\mu \neq 0$: $z_{t+1} = (\lambda z_t - \mu\pi)/(1-\pi)$ with probability $1-\pi$ and $z_{t+1} = \mu$ with probability π.

An important feature of bubbles in the linearized model (5) is that the expected path of the bubbles explodes: $\lim_{s\to\infty} E_t z_{t+s} = \infty$ when $z_t > 0$ and $\lim_{s\to\infty} E_t z_{t+s} = -\infty$ when $z_t < 0$. As discussed in Sect. 1, this feature greatly limits the appeal of the Blanchard (1979) type bubble. Note that, in the Long-Plosser model, the investment/output ratio is bounded by 0 and 1: an infinite investment ratio is not feasible. The linear approximation (on which (5) is based) neglects this constraint. A linear approximation is thus not suitable for studying rational bubbles.

2.2. Stationary bubbles in the non-linear Long-Plosser model

I now show that, by contrast to the linearized model, the **non-linear** Long-Plosser model can produce **stationary** bubbles. Note that (4) holds for any process $\{Z_t\}$ such that

$$\alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1+\varepsilon_{t+1}, \tag{7}$$

where ε_{t+1} is a forecast error with zero conditional mean: $E_t\varepsilon_{t+1}=0$. ε_{t+1} reflects unanticipated changes in Z_{t+1} that are driven by changes in households' expectations about the future path $\{Z_{t+s}\}_{s>1}$. (7) can be written as:

$$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1})\equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1}). \tag{8}$$

Z_{t+1} is strictly increasing and strictly concave in both Z_t and in ε_{t+1}, for $\varepsilon_{t+1}>-1$. The strict concavity reflects decreasing returns and risk aversion. Fig.1 plots Z_{t+1} as a function of Z_t, and that for three values of ε_{t+1}: $\varepsilon_{t+1}=0$ (thick black line), $\varepsilon_{t+1}=0.5$ and $\varepsilon_{t+1}=-0.5$ (thin dashed lines). Throughout this Section, I set α=0.35 and β=0.99, so that $\alpha\beta=0.3465$; these parameter values are standard in quarterly business cycle models.

In a deterministic economy, $\varepsilon_t=0$ holds $\forall t$, and the dynamics of the investment/output ratio obeys thus $Z_{t+1}=\Lambda(Z_t,0)$ (see (8)). Fig. 1 shows that the function $Z_{t+1}=\Lambda(Z_t,0)$ cuts the 45-degree line at two points: $Z_t=Z_{t+1}=\alpha\beta$ and $Z_t=Z_{t+1}=1$. In a deterministic economy, the slope of the mapping from Z_t to Z_{t+1} is $1/(\alpha\beta)$, at the steady state $Z=\alpha\beta$. In a deterministic economy, a realization $Z_t<\alpha\beta$ puts the investment ratio on a trajectory that reaches Z=0 in finite time; after Z=0 has been reached, output and consumption are zero indefinitely. By contrast, a realization $Z_t>\alpha\beta$ initiates a path that converges asymptotically to Z=1 (without ever reaching Z=1), in a deterministic economy.

The main contribution of this paper is to show that there exist stationary bubble equilibria. These bubble equilibria do not converge to Z=0 or Z=1. Thus, consumption and capital are strictly positive in all periods. In what follows, I focus on these stationary (interior) model solutions. When $Z_t<\alpha\beta$, then the law of motion (8) implies that the economy can hit a zero-capital corner solution in subsequent periods. I thus restrict attention to solutions for which $\{Z_\tau\}$ stays forever in the interval $[\alpha\beta,1)$. It is apparent from Fig. 1 that this requires that the support of the distribution of ε_{t+1} has to be bounded from below. (8) implies that when $Z_t\in[\alpha\beta,1)$ holds, then $Z_{t+1}\in[\alpha\beta,1)$ requires $\varepsilon_{t+1}\geq-1+[\alpha\beta/(1-\alpha\beta)]\cdot[1/Z_t-1]\geq-1.$ [4]

For simplicity, and analogy to the Blanchard (1979) bubbles, I assume that ε_{t+1} only takes two values: $-\overline{\varepsilon_t}$ and $\overline{\varepsilon_t}\cdot\pi/(1-\pi)$ with exogenous probabilities π and $1-\pi$, respectively, where $\overline{\varepsilon_t}\in[0,1)$. Z_{t+1} then takes these two values with probabilities π and $1-\pi$:

$$Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon_t}) \text{ and } Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon_t}\pi/(1-\pi)) \text{ with } Z^L_{t+1}\leq Z^H_{t+1}\leq 1. \tag{9}$$

In the spirit of Blanchard (1979), I assume that when an investment 'crash' occurs in period t+1, then the investment/output ratio takes a value that is close to the no-bubble investment/output ratio $\alpha\beta$. Specifically, I postulate that $Z^L_{t+1}=\alpha\beta+\Delta$, where $\Delta>0$ is a small positive constant. A strictly positive value of Δ is needed to generate recurrent bubbles.[5] When we set $Z^L_{t+1}=\alpha\beta+\Delta$, the first equation in (9) pins down $-\overline{\varepsilon_t}$; substitution into the second equation shown in (9) then determines Z^H_{t+1}.

Alternatively, note that under the assumed bubble process with $Z^L_{t+1}=\alpha\beta+\Delta$, the Euler equation (4) can be expressed as

$$\pi H(\alpha\beta+\Delta,Z_t)+(1-\pi)H(Z^H_{t+1},Z_t)=1. \tag{10}$$

For any $Z_t\in[\alpha\beta+\Delta,1)$ there exists a unique value $Z^H_{t+1}\in[\alpha\beta+\Delta,1)$ that solves (10).

Consider an economy that starts in period t=0, with an initial capital stock K_0. Let $u_t\in\{0;1\}$ be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities π and $1-\pi$, respectively. A **bubble equilibrium** is a sequence of investment/output ratios $\{Z_t\}_{t\geq0}$ defined by

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[4] The lower bound of ε_{t+1} is strictly negative if $Z_t>\alpha\beta$, and it is strictly decreasing in Z_t.

[5] Assume that $\Delta=0$ (so that $Z^L_{t+1}=\alpha\beta$) and consider what happens when $Z_t=\alpha\beta$. The first equation shown in (9) then becomes $\alpha\beta\equiv\Lambda(\alpha\beta,-\overline{\varepsilon_t})$ which implies $\overline{\varepsilon_t}=0$, so that $Z^H_{t+1}=Z^L_{t+1}=\alpha\beta$, i.e. Z is (forever) stuck at $\alpha\beta$. Setting $\Delta>0$ rules out that absorbing state.

$Z_0 \in [\alpha\beta + \Delta, 1)$ and $Z_{t+1} = Z^L_{t+1} \equiv \alpha\beta + \Delta$ if $u_{t+1}=0$ and $Z_{t+1} = Z^H_{t+1}$ if $u_{t+1}=1$, for t≥0, where Z^H_{t+1} solves the date t Euler equation (10).

Note that the investment/output ratio in the initial period, Z_0, does not obey the recursion that governs investment ratios in subsequent periods. However, $Z_0 \in [\alpha\beta + \Delta, 1)$ has to hold to ensure that investment/output ratios in all subsequent periods lie in the interval $[\alpha\beta + \Delta, 1)$. Given a sequence $\{Z_t\}_{t\geq 0}$, the path of capital $\{K_{t+1}\}_{t\geq 0}$ can be generated recursively (for the given initial capital stock K_0) using $K_{t+1} = Z_{t+1}\theta_t (K_t)^\alpha$ for t≥0.

I now discuss numerical simulations in which I set $\Delta=0.01$ and $\pi=0.5$. Panel (a) of Fig. 2 plots Z^L_{t+1}, Z^H_{t+1} and $E_t Z_{t+1} = \pi_t Z^L_{t+1} + (1-\pi_t) Z^H_{t+1}$, as functions of Z_t. Also shown in Panel (a) is the value of Z_{t+1} that would obtain in a deterministic economy $(\varepsilon_{t+1}=0)$: $Z_{t+1} = \Lambda(Z_t, 0)$. In the stochastic bubble equilibrium, the investment/output ratio grows between t and t+1 $(Z_{t+1} > Z_t)$ when $\varepsilon_{t+1} = \overline{\varepsilon}_t \cdot \pi/(1-\pi) > 0$; when $\varepsilon_{t+1} = -\overline{\varepsilon}_t$, the investment rate either remains unchanged at $\alpha\beta + \Delta$ (if $Z_t = \alpha\beta + \Delta$), or it drops to $Z_{t+1} = \alpha\beta + \Delta$ (if $Z_t > \alpha\beta + \Delta$).

Fig. 2 shows that Z^H_{t+1} is a steeply increasing function of Z_t. A sequence of positive draws of the forecast error ε thus generates a run of rapid increases in the investment ratio, that is followed by an abrupt contraction in the investment ratio once a negative draw of ε is realized. A sequence of negative forecast errors keeps the investment ratio at the lower bound $\alpha\beta + \Delta$.

The strict concavity of the recursion $Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1})$ with respect to ε_{t+1} (which reflects household risk aversion) implies that $E_t Z_{t+1} < \Lambda(Z_t, 0)$. For any given Z_t, the conditional mean of the date t+1 investment ratio $E_t Z_{t+1}$ is thus strictly below the value of Z_{t+1} that would obtain in a deterministic economy $(\Lambda(Z_t, 0))$.

$E_t Z_{t+1}$ is an increasing and strictly concave function of Z_t: $E_t Z_{t+1} = \zeta(Z_t)$, $\zeta' > 0, \zeta'' < 0$. The graph of $E_t Z_{t+1}$ intersects the 45-degree line at $Z_t = 0.62$. Strict concavity of ζ implies that the unconditional mean of the investment/output ratio $E(Z)$ is smaller than 0.62 (as $E(Z) = E(\zeta(Z)) < \zeta(E(Z))$. The unconditional mean of the investment ratio is $E(Z) = 0.45$. As discussed above, in a deterministic economy, the investment-output ratio would rise steadily and converge to 1, after a value $Z_t > \alpha\beta$ is realized. With stochastic bubbles, by contrast, the investment ratio does not converge to 1; instead, it fluctuates around a mean value that is close to the stationary no-bubbles investment ratio ($\alpha\beta$).

A stochastic bubble implies that the absolute value of the forecast error ε_{t+1} is larger the greater Z_t. Thus, the variance of the forecast error ε_{t+1} is an increasing function of Z_t. Figure 1 shows that the conditional variance of Z_{t+1} is likewise increasing in Z_t. Furthermore, the conditional distribution of Z_{t+1} is left skewed. The left-skewness is likewise increasing in Z_t: the greater the bubble at date t, the bigger the (negative) 'correction' if the bubble bursts in $t+1$.

Panel (b) of Fig. 2 shows representative simulated paths of output, consumption, gross investment (*I*) and of the investment/output ratio (*Z*). In order to assess whether the bubble alone can generate a realistic business cycle, I assume that TFP is constant. The Figure shows that the model generates massive swings in investment and output. During an expanding bubble, the rapid rise in investment is accompanied by a contraction in consumption.

Table 1 reports moments of HP filtered logged time series generated by the model. Line 1 of Panel (a) shows moments for specification I, with probability $\pi_t = 0.5$. Predicted moments are based on a simulation run of 10000 periods.[6] The predicted standard deviation of output is 11.7% which is about five times larger than the historical standard deviation of quarterly GDP in advanced economies. The model-predicted volatility of consumption and investment too is excessive, when compared to the data. The model predicts that output, consumption and investment are serially correlated. However, consumption is predicted to be countercyclical, which is inconsistent with the data.

The model variant above assumes a constant 50% probability that the bubble grows next period. The model predicts smaller, more realistic, fluctuations in real activity occur if we assume that the probability of growth in the bubble falls once the investment/output ratio exceeds a threshold. As an illustration, assume that π_t is very close to unity, for values of the

[6] The initial investment/output ratio is set at $Z_0 = \alpha\beta + \Delta$. Due to stationarity, Z_0 does not affect simulated moments over a long simulation run. The effect of the initial Z_0 on subsequent simulated values vanishes fast.

investment/output ratio greater than 0.36.[7] This threshold is chosen as it generates (more) realistic output volatility. It implies that the investment/output ratio oscillates between these two values: 0.3565 and 0.3916 (see below). Note that the 'High' investment ratio exceeds the 'Low' ratio by about 10%. When the investment/output ratio at date t takes the 'Low' value $Z^L \equiv \alpha\beta + \Delta = 0.3565$, then next period's investment ratio is either 'Low' (Z^L) of 'High' $(Z^H(\alpha\beta+\Delta, 0.5) = 0.3916)$ with 50% probability. If the date t investment ratio is 'High', then the investment ratio falls to the 'Low' value in the next period almost surely. Panel (c) of Figure 2 shows simulated sample paths generated for this model version, and the second Line in Panel (a) of Table 1 reports the corresponding model-predicted business cycle statistics. This model variant produces output fluctuations that are more in line with the data (predicted standard deviation of GDP: 1.33%), however now output, consumption and investment are negatively serially correlated.

2.3. How should policy respond to bubbles?

A state-contingent tax on capital income can eliminate *stochastic* bubbles. Assume that date t capital income is taxed at rate $\tau_{K,t}$ and that tax proceeds are rebated to the household in a lump sum fashion. The household's Euler equation then is: $E_t\beta(C_t/C_{t+1})(1-\tau_{K,t+1})\alpha Y_{t+1}/K_{t+1} = 1$, which implies $\beta(C_t/C_{t+1})(1-\tau_{K,t+1})\alpha Y_{t+1}/K_{t+1} = 1 + \varepsilon_{t+1}$, where ε_{t+1} is a forecast error $(E_t\varepsilon_{t+1} = 0)$. Setting $\tau_{K,t+1} = -\varepsilon_{t+1}$ implies $\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1} = 1$ and thus $Z_{t+1} = 1 - \alpha\beta(1/Z_t - 1)$ holds. This tax policy hence neutralizes stochastic sunspot shocks (however, if Z_0, the investment/output ratio in the initial period t=0, is larger than $\alpha\beta$, then Z will converge to 1).

2.4. Transversality condition

Long and Plosser (1983) assume an infinitely-lived representative household. The competitive equilibrium of the Long-Plosser economy corresponds to the maximum of the household's decision problem. As that decision problem is a well-behaved concave programming problem, its

[7] I set $\pi_t = 0.5$ when $Z_t \in [\alpha\beta + \Delta, 0.36]$ and $\pi_t = 1 - 10^{-100}$ when $Z_t > 0.36$.

solution is unique. The necessary and sufficient optimality conditions of that decision problem are the resource constraint (2), the Euler equation (3) and a transversality condition (TVC) that requires that the value of the capital stock is zero, at infinity: $\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0$. Note that, under the assumptions of the Long-Plosser model, $u'(C_t)K_{t+1}=K_{t+1}/C_t=Z_t/(1-Z_t)$. When $Z_t=\alpha\beta$ holds, then $u'(C_t)K_{t+1}=\alpha\beta/(1-\alpha\beta)$, which shows that the textbook solution $Z_t=\alpha\beta\ \forall t$ satisfies the TVC. Uniqueness of the infinitely-lived household's decision problem implies that any other process $\{Z_t\}$ that is consistent with (2) and (3) has to violate the TVC. This implies that the bubble equilibrium discussed above violates the TVC.[8]

This paper focuses on *stationary* model solutions consistent with the resource constraint and the Euler equation, but it disregards the TVC. The purpose of the paper is to show that stationary rational bubbles can exist in standard non-linear DSGE models. Note that explosive bubbles in linear models (Blanchard (1979)) likewise violate the TVC.

One possible justification for disregarding the TVC is to assume an OLG structure with finitely-lived households. The Appendix presents an OLG structure that has the *same* aggregate resource constraint and the *same* aggregate Euler equation as the Long-Plosser model. Thus equations (1)-(4) continue to hold in that OLG structure, but there is no TVC in the OLG structure. Such an OLG structure provides a motivation for exploring rational bubbles in standard DSGE models. The two key features of the OLG structure are: [9] (I) Efficient risk sharing between periods t and t+1, among all agents who are alive in both periods. (II) Newborn agents receive a wealth endowment such that consumption by newborns represents a time-invariant share of aggregate consumption -- under log utility, this requires that the wealth endowments of newborns is a time-invariant fraction of total wealth across all generations. [10]

[8] Under the bubble process (9), Z_t approaches 1 if a long uninterrupted string of positive draws of the sunspot ε is realized, which entails large positive values of $Z_t/(1-Z_t)$. Although this only happens with a very small probability, it causes the TVC to be violated.

[9] Assumption I is also used by Gali (2018). Assumption II is novel (to the best of my knowledge). Assumptions I and II allow to derive simple non-linear dynamic relations among aggregate variables for the OLG economy. Without these two assumptions, approximate aggregation across generations may be possible, based on linear approximations. The focus of the paper here is on stationary rational bubbles induced by non-linearity. Thus, aggregation based on linear approximations is not useful here.

[10] Assume that the young can appropriate a constant share of total wealth. The wealth endowment of newborn may be provided by bequest, or by a government transfer financed by a (lump sum) tax levied on older generations. In reality, all societies make significant transfers to young generations (e.g., through spending on their health and

Another motivation for disregarding the TVC is that detecting TVC violations may be very difficult, in models that are more complicated than the Long-Plosser model, i.e. in models for which no closed form solution exists (see below). TVC violations may be caused by low-probability events in a distant future. Agents may thus lack the cognitive/computing power to detect deviations from TVC; see discussion in Blanchard and Watson (1982), Lansing (2010) and Ascari et al. (2019). [11]

3. Stationary rational bubbles in an RBC model with incomplete capital depreciation

I next construct an equilibrium with stationary rational bubbles for an RBC model with incomplete capital depreciation and variable labor. It is now assumed that the period utility function is $U(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t)$, $\Psi>0$, where $0\leq L_t\leq 1$ are hours worked. (The numerical methods discussed below allow to handle richer, non-logarithmic, functional forms for period utility.) The household's total time endowment (per period) is normalized to one, so that $1-L_t$ is household leisure.[12] The resource constraint and the production technology are

$$C_t+K_{t+1}=Y_t+(1-\delta)K_t \text{ with } Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{10}$$

where $0<\delta\leq 1$ is the depreciation rate of capital. TFP θ_t is exogenous and follows a stationary AR(1) process. The economy has these efficiency conditions:

$$C_t\Psi/(1-L_t)=(1-\alpha)\theta_t(K_t)^{\alpha}(L_t)^{-\alpha} \quad \text{and} \tag{11}$$

$$E_t\beta\{C_t/C_{t+1}\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha}+1-\delta)=1. \tag{12}$$

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education and through bequests). Wealthy countries make bigger transfers to the young than poor countries. It seems reasonable to assume that the wealth endowment of the young is a (roughly) constant share of total wealth.

[11] Blanchard and Watson (1982) and Ascari et al. (2019) analyze explosive bubbles in linearized models without TVC. Lansing (201) Lansing (2010) disregards the TVC in a non-linear Lucas-style asset pricing models with bubbles, arguing that "agents are forward-looking but not to the extreme degree implied by the transversality condition" (p.1157); Lansing documents the existence of stationary asset price bubbles, when the TVC is dropped. The present paper considers fully-fledged DSGE macro models with endogenous output and capital accumulation. The method for constructing stationary bubble equilibria here is different and more general than that proposed by Lansing (it can be applied to a wide range of macro models).

[12] Due to decreasing returns to capital and a positive capital depreciation rate, the assumed upper bound on hours worked implies that the support of the distribution of capital and output is bounded, in equilibrium, which greatly simplifies the analysis. Some widely used preference specifications (such as $U(C_t,L_t)=\ln(C_t)-\Psi\cdot(L_t)^{\mu}$, $L_t\geq 0$, $\mu>1$) do not impose an upper bound on hours worked. Then the support of the distribution of hours, capital and output may be unbounded, in stationary bubble equilibria, which makes it much harder to analyze (and compute) those equilibria.

(11) indicates that the household's marginal rate of substitution between leisure and consumption is equated to the marginal product of labor, while (12) is the Euler equation.

(11) shows that hours worked L_t are a decreasing function of consumption C_t. Maximum hours worked $L_t=1$ are chosen when consumption is zero. Provided gross investment $I_t \equiv K_{t+1}-(1-\delta)K_t$ does not exceed maximum output (i.e. output with $L_t=1$)

$$I_t \leq \theta_t (K_t)^{\alpha}, \tag{13}$$

equations (10) and (11) uniquely pin down consumption and hours worked as functions of K_{t+1}, K_t, θ_t:

$$C_t=\gamma(K_{t+1},K_t,\theta_t) \text{ and } L_t=\eta(K_{t+1},K_t,\theta_t). \tag{14}$$

Using these expressions to substitute out consumption and labor in the Euler equation gives:

$$E_t[\beta\{\gamma(K_{t+1},K_t,\theta_t)/\gamma(K_{t+2},K_{t+1},\theta_{t+1})\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(\eta(K_{t+2},K_{t+1},\theta_{t+1}))^{1-\alpha}+1-\delta)]=1, \tag{15}$$

which can be written as

$$E_t H(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \tag{16}$$

where the function H maps R_+^5 into R. (The function 'H' in (16) differs from the H function used to denote the Euler equation (4) in Sect. 2).

The model thus boils down to an expectational difference equation in capital. Once a process for capital has been found that is consistent with (16) in all periods, one can use (14) to generate sequences for consumption, hours and output that are consistent with the resource constraint (10) and with the intra-temporal efficiency condition (11). Solving the model amounts, thus, to finding a stochastic process for capital that solves (16).

The conventional "no-bubbles" solution that imposes a TVC can be described by a unique policy function $K_{t+1}=\lambda(K_t,\theta_t)$ (e.g., Schmitt-Grohé and Uribe (2004)). Disregarding the TVC allows to generate stationary model solutions in which agents deviate from that no-bubbles decision rule.

By analogy to the bubble process in the Long-Plosser model (see Sect. 2), I consider equilibria with the property that, in any period *t*, the capital stock K_{t+1} takes one of two values: $K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}$ with exogenous probabilities π and $1-\pi$, respectively, with $K_{t+1}^L=\lambda(K_t,\theta_t)e^{\Delta}$, where Δ is a constant. Δ is set to a positive value close to zero in the simulations reported below. Whether K_{t+1}^L or K_{t+1}^H is realized depends on an exogenous i.i.d. sunspot that is assumed independent of TFP (see below). At date t, agents anticipate that the capital stock set in t+1, K_{t+2} likewise takes one of two values $K_{t+2} \in \{K_{t+2}^L, K_{t+2}^H\}$ with probabilities π and $1-\pi$, respectively, where $K_{t+2}^L = \lambda(K_{t+1}, \theta_{t+1})e^{\Delta}$. The Euler equation between dates t and t+1 (see (16)) can then be written as:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) + (1-\pi)\cdot E_t H(K_{t+2}^H, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1 \qquad \text{for } K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}.$$
(17a)

Consider an economy that starts in period t=0, with an initial capital stock K_0. Let $u_t \in \{0;1\}$ be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities π and $1-\pi$, respectively. A sequence of capital stocks $\{K_t\}_{t\geq 0}$ such that, for all $t\geq 0$, $K_{t+2} = K_{t+2}^L \equiv \lambda(K_{t+1}, \theta_t)e^{\Delta}$ if $u_{t+1}=0$ and $K_{t+2} = K_{t+2}^H$ if $u_{t+1}=1$, where K_{t+2}^H solves (17), is a **'bubble equilibrium'**.

K_1 (the capital stock set in period 0) is not pinned down by the conditions of the bubble equilibrium. Henceforth, I set $K_1 = \lambda(K_0, \theta_1)e^{\Delta}$. (The effect of K_0 and K_1 on endogenous variables in later periods vanishes as time progresses, due to the stationarity of the process).

The capital stock chosen in a 'bust' state at date t+1 (if $u_{t+1}=0$) is: $K_{t+2}^L = \lambda(K_{t+1}, \theta_{t+1})e^{\Delta}$. Thus, K_{t+2}^L depends on θ_{t+1}. I assume that, conditional on date t information, a productivity innovation at t+1 has an equiproportional effect on K_{t+2}^L and K_{t+2}^H. Specifically: $K_{t+2}^H = s_t^H \cdot K_{t+2}^L$, where $s_t^H > 0$ is in the date t information set. Thus, $K_{t+2}^H = s_t^H \cdot \lambda(K_{t+1}, \theta_{t+1})e^{\Delta}$. [13] This greatly simplifies the analysis. Substituting the above formula for K_{t+2}^H into the Euler equation (17a) gives:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) + (1-\pi)\cdot E_t H(s_t^H \cdot \lambda(K_{t+1},\theta_{t+1})e^{\Delta}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1. \quad (17b)$$

s_t^H can be determined by solving this Euler equation.

The trajectory of the capital stock is determined in the following sequence: Given K_0, K_1 the Euler equation (17b) for period t=0 pins down s_t^H. At date t=0, agents expect that the capital

[13] The AR(1) specification of TFP implies $\theta_{t+1} = (\theta_t)^{\rho} \cdot e^{\varepsilon_{t+1}^{\theta}}$. Thus $\partial \ln(K_{t+2}^H)/\partial \varepsilon_{t+1}^{\theta} = \partial \ln(K_{t+2}^L)/\partial \varepsilon_{t+1}^{\theta}$.

stock K_2 chosen in period t=1 will equal $K_2^L=\lambda(K_1,\theta_1)e^{\Delta}$ of $K_2^H=s_0^H\cdot\lambda(K_1,\theta_1)e^{\Delta}$, with probabilities π and 1-π, respectively, where s_0^H solves the date t=0 Euler equation (17b). These expectations (about K_2^L, K_s^H), held at t=0, validate the agents' date t=0 choice of K_1. In t=1, the random sunspot u_1 determines whether K_2 equals K_2^L or K_2^H. Given the realized K_1, K_2, agents expect at t=1 that K_3 will equal $K_3^L=\lambda(K_2,\theta_2)e^{\Delta}$ or $K_3^H=s_1^H\cdot\lambda(K_2,\theta_2)e^{\Delta}$, where s_1^H is determined by the date t=1 Euler equation. This process is repeated in all subsequent periods.

The key feature of this equilibrium is, thus, that agents expect at date t that K_{t+2} will equal $K_{t+2}^L=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$ or $K_{t+2}^H=s_t^H\cdot\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$ with probabilities π and 1-π, respectively. In period t+1, agents are free to select a value of K_{t+2} that differs from K_{t+2}^L or K_{t+2}^H, but they chose not to do so, in equilibrium, because a choice $K_{t+2}\in\{K_{t+2}^L, K_{t+2}^H\}$ is validated by their date t+1 expectation that $K_{t+3}\in\{K_{t+3}^L, K_{t+3}^H\}$.

3.1. Economy with constant TFP

To build intuition, consider first a model variant with constant TFP $\theta_t=\theta\ \forall t$, so that the sunspot is the only source of fluctuations. In the constant TFP economy, I write the no-bubbles policy rule for capital as $K_{t+1}=\lambda(K_t)$, and the Euler equation (16) as $E_t H(K_{t+2},K_{t+1},K_t)=1$.

In a deterministic economy, any deviation from the no-bubbles policy function puts the economy on a trajectory that converges to a zero-consumption and/or zero-capital corner (e.g., Blanchard and Fischer (1989)). The present paper shows that there exist stationary stochastic bubble equilibria that do not converge to zero consumption/capital. With constant TFP, the Euler equation (17) between periods t and t+1 becomes:

$$\pi H(\lambda(K_{t+1})e^{\Delta}, K_{t+1}, K_t) + (1-\pi)\cdot H(K_{t+2}^H, K_{t+1}, K_t) = 1. \tag{18}$$

This equation determines K_{t+2}^H as a function of K_t and K_{t+1}.

As discussed in the Appendix, $\Delta>0$ is needed to generate stationary bubbles. When $\Delta=0$ then bubbles are self-ending.[14] $\Delta<0$ implies that the capital stock can be put on a downward trajectory that leads to a zero capital corner. Throughout the subsequent analysis, I will thus assume $\Delta>0$. As discussed in the Appendix, $\Delta>0$ implies that $K_{t+1}^{L}<K_{t+1}^{H}$ holds, i.e. we can interpret $u_t=0$ and $u_t=1$ as investment 'bust' and 'boom' states, respectively, while π represents the 'bust' probability.

Let $K^{\max}$ be the maximum feasible constant capital stock, $K^{\max}=\theta(K^{\max})^{\alpha}+(1-\delta)K^{\max}$, and let $K_{\Delta}^{\min}$ be the steady state capital stock that would hold if $K_{t+1}=\lambda(K_t)e^{\Delta}$ held each period: $K_{\Delta}^{\min}=\lambda(K_{\Delta}^{\min})e^{\Delta}$. Clearly, $K_{\Delta}^{\min}<K^{\max}$ (for values of Δ close to zero). If the initial capital stock is in the range $K_0\in(K_{\Delta}^{\min},K^{\max})$, then the capital stock stays in that range, in all subsequent periods, when $\Delta\geq 0$ is assumed. An *uninterrupted* infinite sequence of investment booms (driven by an uninterrupted string of u=1 sunspot realizations) would asymptotically drive the capital to the upper bound $K^{\max}$. An *uninterrupted* infinite sequence of investment busts (i.e. a string of u=0 realizations) would asymptotically drive the capital stock to lower bound $K^{\min}$. Of course, such infinite boom or bust runs have zero probability.

3.2. Economy with stochastic TFP

As in the case with constant TFP, we have to set $\Delta>0$ to ensure existence of a stationary bubble equilibrium. Simulations of the RBC model with stochastic TFP discussed below assume the following process for TFP: $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon_{t+1}^{\theta}$, $0\leq\rho<1$, where $\varepsilon_{t+1}^{\theta}$ is a white noise with standard deviation $\sigma_{\theta}>0$. To simplify computations, it is assumed that $\varepsilon_{t+1}^{\theta}$ only takes 2 values with equal probability: $\varepsilon_{t+1}^{\theta}\in\{-\sigma_{\theta},\sigma_{\theta}\}$.

[14] Let Δ=0. Consider a situation with $u_t=0$, so that $K_{t+1}=K_{t+1}^{L}\equiv\lambda(K_t)$. Then (18) is solved by $K_{t+2}^{H}\equiv\lambda(\lambda(K_t)=\lambda(K_{t+1})$, because $H(\lambda(\lambda(K_{t+1})),\lambda(K_t),K_t)=1$. Thus, $K_{v+1}=K_{v+1}^{L}=K_{v+1}^{H}\equiv\lambda(K_v)$ holds $\forall v>t$. Hence, if K_{t+1} equals the value defined by the no-bubbles decision rule, then the agent has to continue sticking to the no-bubbles decision rule in all subsequent periods, and thus the trajectory of the capital stock becomes deterministic.

3.3. Simulation results

I set α=1/3, β=0.99. The capital depreciation rate is set at δ=0.025. The preference parameter Ψ (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at the steady state.[15] In model variants with stochastic TFP, I set the autocorrelation of TFP at ρ=0.979, while the standard deviation of TFP innovations is set at σ_θ=0.0072. Parameters in this range are conventional in quarterly macro models (King and Rebelo, 1999). The no-sunspot policy rule for capital, $\lambda(K_{t+1},\theta_t)$, is approximated using a second-order Taylor expansion.

Table 2 reports simulated business cycle statistics for several model variants. Standard deviations (in %) of GDP (Y), consumption (C), investment (I) and hours worked (L) are reported, as well as correlations of these variables with GDP, autocorrelations and mean values. The reported statistics are based on a simulation run of T=10000 periods.[16] The standard deviations and correlations are median moments computed across rolling windows of 200 periods.[17] By contrast, mean values (of Y,C,I,L) are computed for the whole simulation run (T periods) and expressed as % deviations from the deterministic steady state. The Table also reports the sample mean of the difference between capital income and investment spending (where this difference is normalized by GDP), as well as the fraction of the T periods in which this difference is positive.

3.3.1. Model versions with just bubble shocks

Cols. (1)-(4) of Table 2 pertain to model variants with just bubble shocks (constant TFP). Cols. (5)-(8) assume simultaneous bubble and TFP shocks. Cols. (9)-(10) assume just TFP shocks, without bubbles (the no-bubbles equilibrium is computed using a linear model approximation).

[15] (11) implies that the Frisch labor supply elasticity (LSE) with respect to the real wage rate (marginal product of labor) is $LSE=(1-L)/L$ at the steady state, where L are steady state hours worked. Ψ is set such that $L=0.5$, which implies LSE=1.

[16] For several of the model variants, I also considered simulation runs with T=1000000 periods. The predicted statistics are virtually unchanged when the much longer runs are used.

[17] For each 200-periods window of artificial data, I computed standard deviations and correlation, using logged series that were HP filtered in the respective window. The Table reports median values (across all windows) of these standard deviations and correlations. 200-periods windows of simulated series are used as the historical business cycle statistics reported in Col. 11 of Table 2 pertain to an empirical sample of 200 quarters (see below).

Col. (11) reports historical statistics for the US.[18] Cols. labelled 'Unit Risk Aversion' (or 'Unit RA') assume log utility (see above). Columns labelled 'High RA' assume greater risk aversion for consumption: $U(C_t,L_t)=\ln(C_t-\overline{C})+\Psi\cdot\ln(1-L_t),$ where $\overline{C}$ is a constant that is set at 0.8 times steady state consumption. The 'High RA' preferences imply that consumption has a strictly positive lower bound: $C_t\geq\overline{C}>0.$ In the 'High RA' case, the coefficient of relative risk aversion is 5, at the steady state consumption level (risk aversion is higher for consumption levels below steady state consumption).

All numerical simulations in Table 2 assume Δ=0.001. That value generates standard deviations of real activity that are roughly in line with empirical statistics (higher values of Δ induce greater volatility and a greater unconditional mean of real activity variables). Cols. (1), (3), (5) and (7) assume a bust probability π=0.5, while Cols, (2),(4),(6) and (8) assume π=0.2.

Simulated paths of GDP (continuous black line), consumption (red dashed line), investment (blue dash-dotted line) and hours worked (blue dotted line) are shown in Figure 3. Panel (i) (i=1,..,10) of the Figure assumes the model variant considered in Col. (i) of Table 2. GDP, C and I series shown in Fig. 2 are normalized by steady state GDP; hours worked are normalized by steady state hours.

Cols. (1) of Table 2 assumes unit risk aversion and a bust probability π=0.5. Constant TFP is postulated, so that economic fluctuations are purely driven by the bubble shocks. The predicted standard deviations of output, consumption, investment and hours are 0.49%, 1.08%, 4.29% and 0.74%, respectively. The model-predicted output volatility is about 1/3 of the empirical GDP volatility. Consistent with the data, investment is predicted to me more volatile than output. However, the model predicts that consumption is more volatile than output, which is counterfactual. As in other models driven by investment shocks, the model variant here predicts that consumption is negatively correlated with output; however, the model predicts that investment and hours worked are strongly procyclical, as is consistent with the data. In the model, output, consumption, investment and hours worked are serially correlated, but the predicted autocorrelations (about 0.35) are smaller than the empirical autocorrelations (about 0.85).

[18] Historical standard deviations and correlations (HP filtered logged quarterly series) are taken from King and Rebelo (1999) and pertain to the period 1947Q1-1996Q4. Statistics for capital income minus investment were computed using annual data (1929-1985) reported in Abel et al. (1989).

As pointed out above, the bubble equilibrium implies capital over-accumulation (compared to a no-bubbles equilibrium), i.e. the economy is 'dynamically inefficient' (the TVC is violated). Abel et al. (1989) propose an empirical test of dynamic efficiency. Their key insight is that an economy is dynamically efficient if income accruing to capital (i.e. output minus the wage bill) exceeds investment. Table 2 shows that, for all variants of the bubbles model here, the average (capital income – investment)/GDP ratio is positive and large (the average ratio, 9.12%, is only slightly smaller than the value of that ratio in steady state, 9.59%).[19] In fact, capital income also exceeds investment in close to 100% of all periods. This highlights the difficulty of detecting dynamic inefficiency (as discussed above).

Panel (1) of Figure 3 shows that the bubble equilibrium, with unit risk aversion and π=0.5, generates output, labor hours and investment booms that are relatively infrequent and short-lived. Periods of high investment are also periods of low consumption: in the model, a sudden fall in consumption triggers a rise in labor hours and output. However, in most periods, real activity remains close to (but slightly above) its steady state level. This explains the low predicted autocorrelation of real activity.

A lower bust probability π generates bigger and more persistent 'spikes' in real activity, and thus real activity becomes more volatile and more serially correlated. This is illustrated in Col. (2) of Table 2, where unit risk aversion and π=0.2 are assumed (see also Panel (2) of Figure 3). Output, consumption, investment and hours worked are now excessively volatile, when compared to the data. Consumption, again, is predicted to be more volatile than output.

Model variants with 'high risk aversion (RA)' generate less consumption volatility—those variants capture the fact that consumption is less volatile than output; see Cols. (3) and (4) of Table 2 (and Panels (3) and (4) of Fig. 3), where π=0.5 and π=0.2 are assumed, respectively.

In summary, the model versions with just bubbles shocks considered so far can generate a realistic volatility of real activity and of aggregate demand components. Real activity in the model is serially correlated, but less than in the historical data.

Setting the bust probability at lower values (e.g., π=0.05) generates higher, more realistic serial correlation in real activity but the predicted volatility or real activity becomes too large. A

[19]The steady state (capital income – investment)/GDP ratio is $\alpha r/(\delta+r)$ where $r=(1-\beta)/\beta$ is the steady state interest rate.

lower labor supply elasticity and higher consumption risk aversion are then needed to produce realistic volatilities (results available on request).

3.3.2. Model versions with TFP shocks

The no-bubbles model driven by stochastic TFP shocks underpredicts the volatility of real activity, but it captures the fact that consumption is less volatile than output, while investment is more volatile (see Cols. (9),(10)). In the no-bubbles model version, consumption and investment are pro-cyclical; furthermore, real activity is highly serially correlated, which reflects the high assumed autocorrelation (0.979) of TFP.

The bubble equilibrium with TFP shocks generates fluctuations in real activity that are more volatile than the fluctuations exhibited by the no-bubbles equilibrium (see Fig. 3, Cols. (5)-(8)). In this sense, the bubble equilibrium (with TFP shocks) is closer to the historical data.

Panels (5)-(10) of Fig. 3 show that the effect of bubbles on the simulated series is clearly noticeable (compared to the 'no-bubbles' series with just TFP shocks): the bubbles induce rapid, but short-lived increases in investment, hours worked and output.

4. Stationary rational bubbles in Dellas' (1986) two-country RBC models

I next consider the two-country RBC model proposed by Dellas (1986). This model can be viewed as a two-country version of Long and Plosser (1983) model, in the sense that it also assumes full capital depreciation, log utility and Cobb-Douglas production functions. Like the Long-Plosser model, the Dellas model can be solved in closed form.

Assume a world with two symmetric countries, referred to as Home (H) and Foreign (F). The household of country $i=H,F$ has preferences of the type assumed in the closed economy RBC model of Sect. 3. Thus, her period utility function is: $U(C_{i,t},L_t)=\ln(C_{i,t})+\Psi\cdot\ln(1-L_{i,t})$, $\Psi>0$, where $C_{i,t}$ and $L_{i,t}$ are consumption and hours worked. Each country is specialized in the production of a distinct tradable intermediate good. Country i's intermediate good production function is $Y_{i,t}=\theta_{i,t}(K_{i,t})^{\alpha}(L_{i,t})^{1-\alpha}$, where $Y_{i,t}$, $\theta_{i,t}$, $K_{i,t}$ are the intermediate good output, TFP and capital in country i. Capital and labor are immobile internationally. TFP is exogenous and follows a stationary Markov process. The country i household combines local and imported intermediates into a non-tradable final good, using the Cobb-Douglas production aggregator

$$Z_{i,t}=(y^{i}_{i,t}/\xi)^{\xi}\cdot(y^{j}_{i,t}/(1-\xi))^{1-\xi} \tag{19}$$

where $y^{j}_{i,t}$ is the amount of input *j* used by country *i*. There is local bias in final good production: $\frac{1}{2}<\xi<1$. The final good is used for consumption, $C_{i,t}$, and gross investment, $I_{i,t}$: $Z_{i,t}=C_{i,t}+I_{i,t}$. The law of motion of country *i*'s capital stock is $K_{i,t+1}=I_{i,t}$.

At date *t*, the price of country *i*'s final good $(P_{i,t})$ equals its marginal cost:

$$P_{i,t}=(p_{i,t})^{\xi}\cdot(p_{j,t})^{1-\xi}. \tag{20}$$

where $p_{j,t}$ is the price of intermediate good *j*. Input demands are:

$$y^{i}_{i,t}=\xi\cdot(p_{i,t}/P_{i,t})^{-1}Z_{i,t},\ \ y^{j}_{i,t}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-1}Z_{i,t}\ ,\ \text{ for }\ j\neq i. \tag{21}$$

Market clearing for intermediate goods requires

$$y^{i}_{H,t}+y^{i}_{F,t}=Y_{i,t}\ \text{ for } i=H,F. \tag{22}$$

Country *i*'s terms of trade and real exchange rate are defined as $q_{i,t}\equiv p_{i,t}/p_{j,t}$ and $RER_{i,t}\equiv P_{i,t}/P_{j,t}$, with $i\neq j$, respectively. Thus, increases in $q_{i,t}$ and $RER_{i,t}$ represent an improvement in country *i's* terms of trade, and an *appreciation* of its real exchange rate, respectively.

The model assumes complete international financial markets, so that consumption risk is efficiently shared across countries. In equilibrium, the ratio of Home to Foreign households' marginal utilities of consumption is, thus, proportional to the Home real exchange rate (Kollmann, 1991, 1995; Backus and Smith, 1993). Under log utility, this implies that Home consumption spending is proportional to Foreign consumption spending: $P_{H,t}C_{H,t}=\Lambda\cdot P_{F,t}C_{F,t}$, where Λ is a date- and state-invariant term that reflects the (relative) initial wealth of the two countries. In what follows, I assume that the two countries have the same initial wealth, and I hence set $\Lambda=1$. Thus:

$$P_{H,t}C_{H,t}=P_{F,t}C_{F,t}. \tag{23}$$

Each household equates the marginal rate of substitution between leisure and consumption to the marginal product of labor, expressed in units of consumption, which implies

$$C_{i,t}\Psi/(1-L_{i,t})=(p_{i,t}/P_{i,t})(1-\alpha)\theta_{i,t}(K_{i,t})^{\alpha}(L_{i,t})^{-\alpha}. \tag{24}$$

The capital Euler equation of the country *i* household is:

$$E_t\beta(C_{i,t}/C_{i,t+1})[(p_{i,t+1}/P_{i,t+1})\alpha\theta_{i,t+1}(K_{i,t+1})^{\alpha-1}(L_{i,t+1})^{1-\alpha}+1-\delta]=1\,. \qquad (25)$$

Substitution of the intermediate good demand functions (21) into the market clearing condition for intermediates (22) gives:

$$p_{H,t}Y_{H,t}=\xi\cdot(P_{H,t}C_{H,t}+P_{H,t}K_{H,t+1})+(1-\xi)\cdot(P_{F,t}C_{F,t}+P_{F,t}K_{F,t+1})\,,$$

$$p_{F,t}Y_{F,t}=(1-\xi)\cdot(P_{H,t}C_{H,t}+P_{H,t}K_{H,t+1})+\xi\cdot(P_{F,t}C_{F,t}+P_{F,t}K_{F,t+1})\,. \qquad (26)$$

Using the risk sharing condition (23), we can write (26) as:

$$p_{H,t}Y_{H,t}=P_{H.t}C_{H,t}+\xi\cdot P_{H,t}K_{H,t+1}+(1-\xi)\cdot P_{F,t}K_{F,t+1}\ ,$$

$$p_{F,t}Y_{F,t}=P_{H.t}C_{H,t}+(1-\xi)\cdot P_{H,t}K_{H,t+1}+\xi\cdot P_{F,t}K_{F,t+1}\,. \qquad (27)$$

The labor supply equation (24) can expressed as:

$$p_{i,t}Y_{i,t}/(P_{i,t}C_{i,t})=(\Psi/(1-\alpha))L_{i,t}/(1-L_{i,t}) \quad \text{for } i=H,F. \qquad (28)$$

With full capital depreciation, the Euler equation can be written as:

$$\alpha\beta E_t\{(P_{i,t}C_{i,t})/(P_{i,t+1}C_{i,t+1})\}(p_{i,t+1}Y_{i,t+1})/(P_{i,t}K_{i,t+1})=1 \text{ for } i=H,F\,. \qquad (29)$$

These equations can be expressed in terms of Home and Foreign investment/consumption ratios and nominal output/consumption ratios:

$$\kappa_{i,t}\equiv P_{i,t}K_{i,t+1}/(P_{i,t}C_{i,t}) \quad \text{for } i=H,F; \qquad (30)$$

$$g_{i,t}\equiv p_{i,t}Y_{i,t}/(P_{i,t}C_{i,t}) \quad \text{for } i=H,F. \qquad (31)$$

The market clearing, labor supply and Euler conditions (27), (28) and (29) can be written as

$$g_{H,t}=1+\xi\kappa_{H,t}+(1-\xi)\kappa_{F,t}\,,\quad g_{F,t}=1+(1-\xi)\kappa_{H,t}+\xi\kappa_{F,t}\,, \qquad (32)$$

$$g_{i,t}=(\Psi/(1-\alpha))L_{i,t}/(1-L_{i,t}) \quad \text{for } i=H,F, \qquad (33)$$

$$\text{and} \qquad \alpha\beta E_t g_{i,t+1}=\kappa_{i,t} \quad \text{for } i=H,F. \qquad (34)$$

Using the market clearing conditions (32), we can express the Euler equations (34) as:

$$\alpha\beta\cdot E_t(1+\xi\kappa_{H,t+1}+(1-\xi)\kappa_{F,t+1})=\kappa_{H,t} \ \text{ and } \ \alpha\beta\cdot E_t(1+(1-\xi)\kappa_{H,t+1}+\xi\kappa_{F,t+1})=\kappa_{F,t}\,. \qquad (35)$$

The deterministic steady state value of the Home and Foreign investment/consumption ratios is $\overline{\kappa}=\alpha\beta/(1-\alpha\beta)$. Let $\widetilde{\kappa_{i,t}}\equiv\kappa_{i,t}-\overline{\kappa}$ denote the deviation of $\kappa_{i,t}$ from its steady state value. (35) implies:

$$\alpha\beta\cdot E_t(\xi\widetilde{\kappa_{H,t+1}}+(1-\xi)\widetilde{\kappa_{F,t+1}})=\widetilde{\kappa_{H,t}} \ \text{ and } \ \alpha\beta\cdot E_t((1-\xi)\widetilde{\kappa_{H,t+1}}+\xi\widetilde{\kappa_{F,t+1}})=\widetilde{\kappa_{F,t}}\,. \qquad (36)$$

Therefore, $A\cdot\begin{bmatrix} E_t\widetilde{\kappa_{H,t+1}} \\ E_t\widetilde{\kappa_{F,t+1}} \end{bmatrix}=\begin{bmatrix} \widetilde{\kappa_{H,t}} \\ \widetilde{\kappa_{F,t}} \end{bmatrix}$, where $A\equiv\alpha\beta\cdot\begin{bmatrix} \xi & 1-\xi \\ 1-\xi & \xi \end{bmatrix}$. Hence,

$$\begin{bmatrix} E_t\widetilde{\kappa_{H,t+1}} \\ E_t\widetilde{\kappa_{F,t+1}} \end{bmatrix}=B\cdot\begin{bmatrix} \widetilde{\kappa_{H,t}} \\ \widetilde{\kappa_{F,t}} \end{bmatrix}, \text{ with } B\equiv A^{-1}=\frac{1}{\alpha\beta(2\xi-1)}\cdot\begin{bmatrix} \xi & -(1-\xi) \\ -(1-\xi) & \xi \end{bmatrix}. \tag{37}$$

The eigenvalues of the matrix B are $\lambda_S\equiv 1/(\alpha\beta)$ and $\lambda_D\equiv 1/(\alpha\beta(2\xi-1))$. Both eigenvalues are greater than 1, as $0.5<\xi<1$ Hence, the only non-explosive solution of (37) is given by $\widetilde{\kappa_{i,t}}=0$ and thus $\kappa_{i,t}=\alpha\beta/(1-\alpha\beta)$, for $i=H,F$. Dellas (1986) focuses on this 'no-bubble' solution.

4.1. Bubble equilibria in the Dellas model

In what follows, I will study bubble equilibria with $\widetilde{\kappa_{i,t}}\neq 0$. As for previous models discussed in this paper, I focus on 'interior' equilibria for which consumption, capital and output are strictly positive for all dates and states of the world. It can be seen immediately from (30)-(35) that an interior equilibrium requires that the Home and Foreign investment/consumption ratios are strictly positive for all dates and states.

Any strictly positive process for Home and Foreign investment/consumption ratios that satisfies the Euler equations (35),(36) has to be such that

$$\widetilde{\kappa_{H,t}}=\widetilde{\kappa_{F,t}}\geq 0 \ \forall t. \tag{38}$$

Thus, the bubbly investment/consumption ratio has to be always at least as large as the steady state ratio; as in the closed economy models discussed previously, the bubble equilibria in the two-country world exhibit capital over-accumulation. Also, a strictly positive bubble process has to be **identical** across the two countries. To understand this, let $S_t\equiv\widetilde{\kappa_{H,t}}+\widetilde{\kappa_{F,t}}$ and $D_t\equiv\widetilde{\kappa_{H,t}}-\widetilde{\kappa_{F,t}}$ respectively denote the sum and the difference of the two countries' investment/consumption ratios, expressed as deviations from steady state. (37) implies $E_tS_{t+1}=\lambda_S\cdot S_t$ and $E_tD_{t+1}=\lambda_D\cdot D_t$, where λ_S and λ_D are the eigenvalues of the matrix B (see (37)). Note that $\widetilde{\kappa_{H,t}}=\frac{1}{2}\cdot(D_t+S_t)$ and $\widetilde{\kappa_{F,t}}=\frac{1}{2}\cdot(S_t-D_t)$. Hence,

$$E_t\widetilde{\kappa_{H,t+s}}=\tfrac{1}{2}\cdot(\lambda_S)^s\{S_t+(1/(2\xi-1))^sD_t\} \text{ and } E_t\widetilde{\kappa_{F,t+s}}=\tfrac{1}{2}(\lambda_S)^s\{S_t-(1/(2\xi-1))^sD_t\}, \tag{39}$$

where I use the fact that $\lambda_D = \lambda_S/(2\xi - 1)$. $0.5 < \xi < 1$ implies $1/(2\xi - 1) > 1$ and thus $\lambda_D > \lambda_S$. A necessary condition for non-negativity of $\kappa_{H,\tau}$ and $\kappa_{F,\tau}$ in all future dates and states $\tau \geq t$ is $D_t = 0$ and $S_t \geq 0$ which implies (38). [20]

What explains why only **global** bubbles are possible? Intuitively, an investment bubble that occurs solely in the Home country $(\widetilde{\kappa_{H,t}} > 0)$ would trigger a growing Home trade deficit, due to growing intermediate imports by Home, fueled by the boom in Home investment. This would put Foreign investment on a downward trajectory. If the Home bubble lasts sufficiently long, the Foreign capital stock would ultimately reach zero.

More formally, we can note from the Euler equation (36) that if $\widetilde{\kappa_{H,t}}, E_t \widetilde{\kappa_{H,t+1}} > 0$ holds, then $\widetilde{\kappa_{F,t}} = E_t \widetilde{\kappa_{F,t+1}} = 0$ is impossible. Thus a bubble cannot occur just in country H.

The Euler equation prescribe that country i's country's investment/consumption ratio at date t equals the country's expected date t+1 output/consumption ratio multiplied by the factor $0 < \alpha\beta < 1$. The future output/consumption ratio equals 1 plus a weighted average of future domestic and foreign investment/consumption ratios, with weights ξ and 1-ξ, respectively (see (32)). If Home and Foreign had identical final good technologies (zero local spending bias: $\xi = 1/2$), the date t investment/consumption ratio would thus be identical across countries, irrespective of the expected date t+1 investment/consumption ratios. With a local spending bias ($1/2 < \xi < 1$), the difference between Home and Foreign investment/consumption ratios at date t is smaller (in absolute value) than the expected cross-country difference at t+1 (as $1/\lambda_D < 1$). Any difference between domestic and foreign investment/consumption ratios at date t $(D_t \neq 0)$ would trigger a larger expected difference in period t+1; thus, the expected cross-country difference would explode, and that at a faster rate than the sum of these two-country's investment/consumption ratios (as $\lambda_D > \lambda_S$). This would induce potential violations of the non-negativity constraint in future periods $\tau > t$.

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[20] $D_t \neq 0$ would imply $\lim_{s\to\infty} E_t \widetilde{\kappa_{H,t+s}} = -\infty$ or $\lim_{s\to\infty} E_t \widetilde{\kappa_{F,t+s}} = -\infty$. With strictly positive probability, $\kappa_{H,\tau}$ or $\kappa_{F,\tau}$ would thus be **negative** at some date(s) $\tau \geq t$. Setting $D_t = 0$ in (39) shows that $S_t < 0$ would imply $\lim_{s\to\infty} E_t \widetilde{\kappa_{H,t+s}} = -\infty$ and $\lim_{s\to\infty} E_t \widetilde{\kappa_{F,t+s}} = -\infty$, so that $\kappa_{H,\tau} < 0$ or $\kappa_{F,\tau} < 0$ with positive probability at some date(s) $\tau \geq t$.

The subsequent discussion thus assumes that (38) holds. Let $\kappa_t = \kappa_{H,t} = \kappa_{F,t}$ denote the (common) investment/consumption ratio in both countries, and let $\widetilde{\kappa_t} \equiv \kappa_t - \overline{\kappa}$ be its deviation from the steady state ratio κ. (36) implies

$$\alpha\beta E_t \widetilde{\kappa_{t+1}} = \widetilde{\kappa_t}. \tag{40}$$

By analogy to the bubble processes discussed in earlier Sections, assume that $\widetilde{\kappa_{t+1}}$ takes two values: $\widetilde{\kappa_{t+1}} \in \{\Delta, \widetilde{\kappa_{t+1}^H}\}$ with exogenous probabilities π and $1-\pi$, respectively, and $\Delta>0$. Consider a world economy that starts in period t=0, with initial capital stocks $K_{H,0}, K_{F,0}$. Let $u_t \in \{0;1\}$ be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities π and $1-\pi$, respectively. An (interior) **bubble equilibrium** is a sequence of investment/consumption ratios $\{\widetilde{\kappa_t}\}_{t\geq 0}$ such that $\widetilde{\kappa_{t+1}} = \Delta$ if $u_{t+1}=0$ and $\kappa_{t+1} = \widetilde{\kappa_{t+1}^H}$ if $u_{t+1}=1$, for t≥0, where $\widetilde{\kappa_{t+1}^H}$ solves the date t Euler equation (40). Note that (40) implies $\alpha\beta\{\pi\Delta + (1-\pi)\widetilde{\kappa_{t+1}^H}\} = \widetilde{\kappa_t}$, so that $\widetilde{\kappa_{t+1}^H} = (\widetilde{\kappa_t} - \alpha\beta\pi\Delta)/(\alpha\beta(1-\pi))$. If $\widetilde{\kappa_t} \geq \Delta$ holds, then $\widetilde{\kappa_{t+1}^H} > \widetilde{\kappa_t}$.

The investment/consumption ratio in the initial period, $\widetilde{\kappa_0}$, does not obey the recursion that governs investment/consumption ratios in subsequent periods. However, $\widetilde{\kappa_0} \geq \Delta$ has to hold to ensure that $\widetilde{\kappa_t} \geq \Delta$ holds $\forall t>0$. [21]

Given the κ_t process, one can solve for the real exchange rate, hours worked, consumption, investment and output in both countries, using the static equilibrium conditions (26)-(33). (33) implies that labor hours in country i are determined by the output/consumption ratio $g_{i,t}$. In equilibrium, both countries have the same (nominal) output/consumption ratio, as $g_{i,t} = 1+\kappa_t$ for $i=H,F$. Equilibrium hours worked are, thus, identical across countries:

$$L_{i,t} = L_t \equiv (1+\kappa_t)/\{1+\kappa_t + \Psi/(1-\alpha)\} \text{ for } i=H,F. \tag{41}$$

and country i output is:

$$Y_{i,t} = \theta_{i,t}(K_{i,t})^{\alpha}(L_t)^{1-\alpha}. \tag{42}$$

[21] Note that $\kappa_t = \widetilde{\kappa_t} + \alpha\beta/(1-\alpha\beta)$. As in the models discussed in earlier Sections, $\Delta>0$ is needed to ensure a strictly positive recurrent bubble. $\Delta=0$ would imply that the bubble is self-ending ($\widetilde{\kappa_t}=0$ would imply $\widetilde{\kappa_\tau}=0$ $\forall$ τ>t.)

$\kappa_{H,t}=\kappa_{F,t}$ also implies that investment and output, valued at market prices, are equated across countries: $P_{H,t}K_{H,t+1}=P_{F,t}K_{F,t+1}$ and $p_{H,t}Y_{H,t}=p_{F,t}Y_{F,t}$ (from (23), (30) and (31)). Because consumption spending is likewise equated across countries (see (23)), net exports are zero, in equilibrium. Country i's terms of trade equal the inverse of i's relative output: $q_{i,t}\equiv p_{i,t}/p_{j,t}=Y_{j,t}/Y_{i,t}$ with $j\neq i$. The real exchange rate is $RER_{i,t}=(q_{i,t})^{2\xi-1}$. (31) implies $C_{i,t}=(1/(1+\kappa_t))(p_{i,t}/P_{i,t})Y_{i,t}$. Note that $p_{i,t}/P_{i,t}=(q_{i,t})^{1-\xi}=(Y_{j,t}/Y_{i,t})^{1-\xi}$ with $j\neq i$. Thus:

$$C_{H,t}=(1/(1+\kappa_t))(Y_{H,t})^{\xi}(Y_{F,t})^{1-\xi} \quad \text{and} \quad C_{F,t}=(1/(1+\kappa_t))(Y_{H,t})^{1-\xi}(Y_{F,t})^{\xi}\,. \tag{43}$$

Finally, note from (30) that $K_{i,t+1}=\kappa_t C_{i,t}$. Therefore, date t investment is given by:

$$K_{H,t+1}=(\kappa_t/(1+\kappa_t))(Y_{H,t})^{\xi}(Y_{F,t})^{1-\xi} \quad \text{and} \quad K_{F,t+1}=(\kappa_t/(1+\kappa_t))(Y_{H,t})^{1-\xi}(Y_{F,t})^{\xi}\,. \tag{44}$$

Hours worked, investment and output at date t are increasing functions of κ_t, while consumption is decreasing in κ_t.

(44) implies that logged capital follows a stable vector autoregression:

$$\begin{bmatrix}\ln(K_{H,t+1})\\ \ln(K_{F,t+1})\end{bmatrix}=\begin{bmatrix}\xi\alpha & (1-\xi)\alpha\\ (1-\xi)\alpha & \xi\alpha\end{bmatrix}\cdot\begin{bmatrix}\ln(K_{H,t})\\ \ln(K_{F,t})\end{bmatrix}+\begin{bmatrix}\omega_H(\theta_{H,t},\theta_{F,t},\kappa_t)\\ \omega_F(\theta_{H,t},\theta_{F,t},\kappa_t)\end{bmatrix}, \tag{45}$$

where $\omega_i(\theta_{H,t},\theta_{F,t},\kappa_t)\equiv(\kappa_t/(1+\kappa_t))(L_t)^{1-\alpha}(\theta_{i,t})^{\xi}(\theta_{j,t})^{1-\xi}$ with $j\neq i$ is a function of hours worked, of TFP in the two countries, and of κ_t. Note that $\kappa_t/(1+\kappa_t)$ is strictly positive and bounded: $\Delta/(1+\Delta)\leq\kappa_t/(1+\kappa_t)<1$. Hours too are strictly positive and bounded: $(1+\Delta)/\{1+\Delta+\Psi/(1-\alpha)\}\leq L_t<1$. Assume that the TFP process is strictly positive and bounded (as in Sect. 3.3.2). Then $\omega_{H,t}$ and $\omega_{F,t}$ are strictly positive and bounded. The eigenvalues of the autoregressive matrix of the law of motion of capital (α and $\alpha(2\xi-1)$) are smaller than 1 in absolute value; thus, the capital stock in each country too is strictly positive and bounded, and so are output, consumption and the terms of trade.

In a deterministic economy, any deviation of the investment/consumption ratio from its steady state value $\alpha\beta/(1-\alpha\beta)$ puts the capital stock on a divergent trajectory that converges to the maximum feasible capital stock. By contrast, the stochastic bubble equilibrium considered here does not entail convergence to the maximum capital stock.

4.2. Simulation results

Table 3 reports simulated business statistics for the Dellas two-country model with bubbles. The capital share, and the subjective discount rate are respectively set at α=1/3 and β=0.99. The share of spending devoted to domestic intermediates is set at ξ=0.9.[22] I set the bust probability at π=0.5. Δ is set at 2.22%, as this implies that, in a bust state the ratio of capital spending divided by nominal GDP $Z_{i,t} \equiv P_{i,t}K_{i,t+1}/(p_{i,t}Y_{i,t})$ is 1% above its steady state value $\alpha\beta$.[23] Versions of the model with TFP shocks assume that Home and Foreign TFP follow the (symmetric) autoregressive process that Backus et al. (1994) estimated using quarterly TFP series for the US and an aggregate of European economies:

$$\begin{bmatrix} \ln\theta_{H,t+1} \\ \ln\theta_{F,t+1} \end{bmatrix} = \begin{bmatrix} .906 & .088 \\ .088 & .906 \end{bmatrix} \cdot \begin{bmatrix} \ln\theta_{H,t} \\ \ln\theta_{F,t} \end{bmatrix} + \begin{bmatrix} \varepsilon^{\theta}_{H,t+1} \\ \varepsilon^{\theta}_{F,t+1} \end{bmatrix}, \tag{46}$$

where $\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}$ are white noises with $Std(\varepsilon^{\theta}_{H,t+1})=Std(\varepsilon^{\theta}_{F,t+1})=0.852\%$ and $Corr(\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1})=0.258$. Thus, productivity is a highly persistent process, and there are delayed positive cross-country spillovers (positive off-diagonal elements of the autoregressive matrix); also, productivity innovations are positively correlated across countries.

Col. 1 of Table 3 considers a version of the model with just bubble shocks (while TFP is constant). Col. 2 assumes joint bubble and TFP shocks, while Col. 3 assumes just TFP shocks (no bubble: $\kappa_t=\kappa \ \forall t$).

In the Dellas model with just bubble shocks, output, consumption, investment and hours are identical across the two countries (see Col. 1). The dynamics of these variables corresponds, thus, to that predicted by a closed economy model Long-Plosser model. Like the Long-Plosser model, the Dellas model with bubbles predicts excessive fluctuations of real activity;

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[22] This calibration is consistent with the fact that the mean US trade share (0.5×(imports+exports)/GDP) was 10% in the period 1973-2013.

[23] Note that $Z_{i,t}=\kappa_t/(1+\kappa_t)$. Hence, $Z_{H,t}=Z_{F,t}$. The assumption that $\kappa_t = \kappa + 0.0222$ holds in a bust ensures that the bust value of $Z_{i,t}$ exceeds its steady state value by 0.01. This parallels the calibration of a bust in the closed economy Long-Plosser model (see Sect. 2).

consumption and investment are predicted to be more volatile than output.[24] Because of the predicted perfect correlation of Home and Foreign output, the terms of trade and the real exchange rate are constant, when there are just bubble shocks.

The variant of the Dellas model with just TFP shocks (no bubbles) generates realistic output and consumption variances (see Col. 3). However, investment, hours worked and the real exchange rate are less volatile than in the data (this model variant predicts that hours are constant). The model variant with just TFP shocks generates fluctuations in output, consumption and investment that are positively correlated across countries. The predicted cross-country correlation of output (0.37) is smaller than the empirical correlation (0.52), while predicted cross-country correlations of consumption and investment (0.55) are higher than in the data.

The model variant with simultaneous bubble shocks and TFP shocks generates fluctuations that are dominated by the bubble shocks. For example, with the joint shocks, real activity is almost perfectly correlated across countries. However, the presence of TFP shocks implies that the real exchange rate shows non-negligible fluctuations.

4.3. Financial autarky

Country-specific bubbles can arise when there are impediments to international capital flows. Consider, for example, a variant of the Dellas model with financial autarky, so that net exports are constrained to be zero (balanced trade). Then the budget constraint of the country *i* household is

$$p_{i,t}Y_{i,t}=P_{i,t}(C_{i,t}+K_{i,t+1}), \tag{47}$$

i.e. the value of a country's intermediate good output equals the value of its consumption and investment spending. Under financial autarky, we thus have (from (47)):

$$g_{i,t}=1+\kappa_{i,t}. \tag{48}$$

Substituting (47) into the market clearing conditions (26) for the Home and Foreign intermediate goods gives: $p_{i,t}Y_{i,t}=\xi\cdot p_{i,t}Y_{i,t}+(1-\xi)\cdot p_{j,t}Y_{j,t}$, $i\neq j$. Thus: $p_{i,t}Y_{i,t}=p_{j,t}Y_{j,t}$ holds, as is the case under complete markets. The labor supply condition (28) and the Euler equation (29) continue to hold under financial autarky, and hence (33) and (34) likewise continue to hold:

[24] The version of the Dellas model considered in Table 3 assumes endogenous labor. Hours worked respond to bubble shocks. This explains why real activity is more volatile than in the Long-Plosser model with fixed labor considered in Table 1.

$$g_{i,t}=(\Psi/(1-\alpha))L_{i,t}/(1-L_{i,t}) \quad \text{and} \quad \alpha\beta E_t g_{i,t+1}=\kappa_{i,t} \quad \text{for } i=H,F.$$

(48) implies that, under financial autarky, labor hours are given by

$$L_{i,t}=(1+\kappa_{i,t})/\{1+\kappa_{i,t}+\Psi/(1-\alpha)\}, \tag{49}$$

while the Euler equation can be expressed as

$$\alpha\beta E_t(1+\kappa_{i,t+1})=\kappa_{i,t}. \tag{50}$$

(47) implies $C_{i,t}=(1/(1+\kappa_{i,t}))(p_{i,t}/P_{i,t})Y_{i,t}$. As $p_{i,t}/P_{i,t}=(p_{i,t}/p_{j,t})^{1-\xi}=(Y_{j,t}/Y_{i,t})^{1-\xi}$, $j\neq i$ we find

$$C_{i,t}=(1/(1+\kappa_{i,t}))(Y_{i,t})^{\xi}(Y_{j,t})^{1-\xi} \quad \text{and} \quad K_{i,t+1}=(\kappa_{i,t}/(1+\kappa_{i,t}))(Y_{i,t})^{\xi}(Y_{j,t})^{1-\xi}. \tag{51}$$

Note that $C_{H,t}/C_{F,t}=((1+\kappa_{F,t})/(1+\kappa_{H,t}))P_{F,t}/P_{H,t}$. Thus,

$$P_{H,t}C_{H,t}=((1+\kappa_{F,t})/(1+\kappa_{H,t}))P_{F,t}C_{F,t}. \tag{52}$$

Recall that the complete markets model too implies that trade is balanced, in equilibrium. (By contrast, balanced trade is a c*onstraint* under financial autarky.) If $\kappa_{H,t}=\kappa_{F,t}$, then bubble equilibria under financial autarky are observationally equivalent to bubble equilibria under complete markets. (52) shows that, in that case, consumption spending under financial autarky is equated across countries, i.e. the international risk sharing condition (23) continues to hold.

Under financial autarky, $\kappa_{i,t}$ solely has to satisfy the country *i* Euler equation (50); thus, country-specific bubbles are possible, $\kappa_{H,t}\neq\kappa_{F,t}$ (By contrast, under complete markets, $\kappa_{i,t}$ also enters the Euler equation of country *j*, and thus the two countries' bubbles are jointly determined—as shown above, this restriction implies that country-specific bubbles are impossible, when markets are complete.) Local bubbles, under financial autarky, induce violations of the international risk sharing condition (see (52)).

5. Rational bubbles in a two-country RBC model with incomplete capital depreciation

This Section discusses a two-country RBC model with incomplete capital depreciation, non-unitary risk aversion and a CES final good aggregator. As in the Dellas model, complete financial markets are assumed. This richer model version cannot be solved in closed form.

The period utility now is: $U(C_{i,t},L_t)=\ln(C_{i,t}-\overline{C})+\Psi\cdot\ln(1-L_{i,t})$, with $\overline{C}\geq 0$. The country *i* final good production function is $Z_{i,t}=[\xi^{1/\phi}\cdot(y_{i,t}^{i})^{(\phi-1)/\phi}+(1-\xi)^{1/\phi}\cdot(y_{i,t}^{j})^{(\phi-1)/\phi}]^{\phi/(\phi-1)}$, $j\neq i$, where ϕ is the substitution elasticity between domestic and imported goods. The price of country i's final good $(P_{i,t})$ now is $P_{i,t}=[\xi\cdot(p_{i,t})^{1-\phi}+(1-\xi)\cdot(p_{j,t})^{1-\phi}]^{1/(1-\phi)}$, $j\neq i$, and input demands are $y_{i,t}^{i}=\xi\cdot(p_{i,t}/P_{i,t})^{-\phi}Z_{i,t}$, $y_{i,t}^{j}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-\phi}Z_{i,t}$. The law of motion of country i's capital stock is $K_{i,t+1}=(1-\delta)K_{i,t}+I_{i,t}$, where $0<\delta\leq 1$ is the capital depreciation rate.

The *static* equilibrium conditions (i.e. the market clearing condition and the labor supply equation) allow to express date t consumption, hours worked and terms of trade $C_{i,t},L_{i,t},q_{i,t}$ for i=H,F as functions of both countries' date t productivity and capital stocks in t and t+1, $K_{i,t},K_{i,t+1},\theta_{i,t}$ for i=H,F. Substituting these functions into the two countries' capital Euler equations (25) allows to express these Euler equations as expectational difference equations in terms of Home and Foreign capital stocks:

$$E_t H_i(\overrightarrow{K_{t+2}},\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})=1 \quad \text{for } i=H,F, \tag{53}$$

where $\overrightarrow{K_t}\equiv(K_{H,t};K_{F,t})$ and $\overrightarrow{\theta_t}\equiv(\theta_{H,t};\theta_{F,t})$ are vectors of Home and Foreign capital and TFP, respectively. The function H_i maps R_{+}^{10} into R.

The no-bubble solution of the two-country model is described by policy functions $K_{i,t+1}=\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})$ that map date t capital and TFP into capital at date t+1.

The specification of the bubble process parallels the specification used in the closed economy RBC model (see Sect. 3). It is, thus, assumed that $K_{i,t+1}$ takes two possible values: $K_{i,t+1}\in\{K_{i,t+1}^{L},K_{i,t+1}^{H}\}$, where $K_{i,t+1}^{L}=\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})\cdot e^{\Delta}$, with $\Delta>0$. As in the Dellas model (with complete financial markets), the investment bubble has to be perfectly synchronized across countries.

Consider an economy that starts at date t=0, with initial capital stocks $K_{H,0},K_{F,0}$. Let $u_t\in\{0;1\}$ be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities π and $1-\pi$, respectively. A **'bubble equilibrium'** is a sequence of capital stocks $\{K_{H,t},K_{F,t}\}_{t\geq 0}$ such that, for all t≥0,

(a) $K_{i,t+1}^{L}=\lambda_i(\overrightarrow{K_{H,t}},\overrightarrow{\theta_{H,t}})\cdot e^{\Delta}$ for i=H,F if $u_{t+1}=0$;

(b) $K_{i,t+2}=K_{i,t+2}^{H}$ i=H,F, if $u_{t+1}=1$, where $K_{H,t+2}^{H},K_{F,t+2}^{H}$ solves (53).

By analogy with the specification in Sect. 3.2, I assume that, conditional on date t information, Home and Foreign productivity innovations at t+1 have equiproportional effects on $K_{i,t+2}^{L}$ and $K_{i,t+2}^{H}$. Specifically: $K_{i,t+2}^{H}=s_{i,t}^{H}\cdot K_{i,t+2}^{L}$, where $s_{i,t}^{H}>0$ is in the date t information set. This greatly simplifies the analysis, as it allows to write the Euler equations (53) as:

$$\pi E_t H_i(\lambda_H(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},\lambda_F(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})+$$

$$(1-\pi)E_t H_i(s_{H,t}^{H}\cdot\lambda_H(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},s_{F,t}^{H}\cdot\lambda_F(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})e^{\Delta},\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})=1 \quad \text{for } i=H,F.$$

Given the date t+1 capital stocks selected by the realization of the sunspot, the date t Euler equations of both countries only feature two endogenous variables chosen in period: $s_{H,t}^{H}$ and $s_{F,t}^{H}$. Thus, simulating the bubble equilibrium boils down to solving two equations in two unknowns in each period.

5.1. Simulation results

As in previous models, I set $\alpha=1/3$, $\beta=0.99$. The capital depreciation rate is set at $\delta=0.025$. The preference parameter Ψ (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at the steady state. As in the calibration of the Dellas model, I set the local spending bias parameter at $\xi=0.9$. The substitution elasticity between domestic and imported intermediates is set at $\phi=1.5$; that value is widely used in the International RBC literature, which is consistent with estimated price elasticities of aggregate trade flows (e.g., Backus et al. (1994)). Model versions with TFP shocks assume that Home and Foreign TFP obey the vector autoregression (46). The parameters of the bubble process are identical to the bubble process in the closed economy RBC model. I consider two values of the bust probability: π=0.2 and π=0.5. Δ is set at Δ=0.001. The no-sunspot decision rules for capital, $\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})$ for $i=H,F$ are again approximated using a second-order Taylor expansion.

Predicted business cycle statistics generated by the two-country RBC model with incomplete capital depreciation are shown in Table 4. Cols. labelled 'Unit Risk Aversion' (or 'Unit RA') assume a log utility function (i.e. minimum consumption is set at $\overline{C}=0$). In Cols. labelled 'High RA', $\overline{C}$ is set at 0.8 times steady state consumption.

Cols. (9) and (10) of Table 4 show simulated business cycle moments for model variants driven just by TFP shocks, i.e. without bubbles. The results confirm findings that are well-know from the International RBC literature (e.g., Backus et al. (1994), Kollmann (1996)): a complete markets model driven just by TFP shocks captures well the volatilities of output and investment, but it underpredicts the volatility of the real exchange rate. The model reproduces the fact that net exports are countercyclical. However, the model-predicted cross-country correlations of output and investment are markedly lower than the corresponding historical correlations. By contrast, the model predicts that consumption is almost perfectly correlated across countries. The low predicted cross-country correlation of output reflects the fact that, with complete financial markets, a positive shock to Home productivity raises Foreign consumption, which reduces Foreign labor supply, and thus Foreign output falls, on impact (while Home output increases).[25]

Simulated business cycle statistics for model variants with just bubble shocks (constant TFP) are reported in Cols. (1)-(4) of Table 4. Standard deviations, correlations with domestic GDP and autocorrelations are *identical* to the corresponding statistics (with just bubble shocks) for the closed economy RBC model (see Col. (1)-(4) of Table 2). This is due to the fact that, in the two-country model with complete markets, bubbles are perfectly correlated across countries; with just bubble shocks, real activity is thus perfectly correlated across countries, the terms of trade are constant and net exports are zero. The volatility of real activity induced by bubble shocks is broadly comparable to the volatility generated by TFP shocks, but the implied volatility of hours worked is higher under bubble shocks.

Simulated business statistic under joint bubbles and TFP shocks are shown in Cols. (5)-(8) of Table 4. With joint shocks, the predicted volatilities of endogenous variables are higher, and thus closer to the data, then with just TFP shocks. The model with joint bubble and TFP shocks is especially successful at matching the positive empirical cross-country correlations of output, investment and hours worked, and the counter-cyclical trade balance; however the predicted cross-country consumption correlation is too high, when compared to the data.

Fig. 4 shows simulated sample paths for the model version with 'High Risk Aversion' and a bust probability π=0.2. Panels (1),(2) and (3) respectively show results for cases with just bubble shocks; with joint bubble and TFP shocks; and with just TFP shocks. With just TFP

[25] The Dellas model with just TFP shocks (no bubbles) generates higher cross-country output correlations (see above) because, in that model, hours worked are constant (in the no-bubbles equilibrium).

shocks (no bubbles), a negative cross-country correlation of high-frequency output and investment fluctuations is clearly discernible (see Panel (3)). Bubble shocks induce relatively widely spaced output and investment booms that are perfectly correlated across countries (Panel (1)). In the setting with joint bubble and TFP shocks, output and investment are markedly more synchronized across countries than when there are just TFP shocks.

6. Conclusion

Linearized Dynamic Stochastic General Equilibrium (DSGE) models with a unique stable solution are the workhorses of modern macroeconomics. This paper shows that stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions. In the sunspot equilibria considered here, the economy may temporarily diverge from the no-sunspots allocation, before abruptly reverting towards that allocation. In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity. Numerical simulations suggest that non-linear DSGE models driven solely by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts. This paper analyzed bubbles in both closed and open economies. A key finding for a two-country model is that, with integrated financial markets, investment bubbles have to be perfectly correlated across countries. Global bubbles may help to explain the synchronization of international business cycles. Country-specific bubbles can only arise when there are impediments to international capital flows.

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Appendix 1: OLG model with same aggregate Euler equation as a model with an infinitely-lived representative agent

This Appendix shows that an economy inhabited by overlapping generations (OLG) of finitely-lived agents can have the same aggregate equations—with the exception of the transversality condition (TVC)--as an economy with an infinitely lived representative agent. Here, this point is made for the Long-Plosser model discussed in Sect.2. It is assumed that the economy has the same aggregate production function and the same aggregate resource constraint as the corresponding representative agent economy.

The two key assumptions that deliver this result are: I. Efficient risk sharing between periods t and t+1, among all agents who are alive in both periods. II. Newborn agents receive a wealth endowment such that consumption by newborns represents a time-invariant share of aggregate consumption. Under log utility, this requires that newborn agents receive a wealth endowment that is a time-invariant share of total wealth.

Assume that agents live N<∞ periods. A measure 1 of agents is born each period. Thus, a fraction 1/N of the population is aged n=1,…,N. All members of the same age cohort are identical. All agents have log utility and the same subjective discount factor, β. Let $c_{i,t}$ denote the date t consumption of agents who are in the i-th period of their life ('generation i') at date t. The expected life-time utility of the generation born at date t is, thus, $E_t \sum_{s=0}^{N-1} \beta^s \ln(c_{i+s,t+s})$. Aggregate consumption at date t is $C_t = \sum_{i=1}^{N} c_{i,t}$. Assume that there exists a market at date t in which a complete set of one-period claims with state-continent date t+1 payouts is traded. This implies that, in equilibrium, the consumption growth rate between t and t+1 is equated across all agents who are alive in both periods (risk sharing):

$$c_{i+1,t+1}/c_{i,t} = c_{2,t+1}/c_{1,t} \text{ for i=1,..,N-1.} \quad \text{(A.1)}$$

Let $\lambda_{i,t} \equiv c_{i,t}/C_t$ denote the ratio of generation i's consumption divided by aggregate consumption, in period t. I refer to $\lambda_{i,t}$ as the 'consumption share' of generation i, in period t. (A.1) implies

$$\lambda_{i+1,t+1}/\lambda_{i,t} = \lambda_{2,t+1}/\lambda_{1,t} \text{ for i=1,..,N-1.} \quad \text{(A.2)}$$

(A.2) and the adding up constraint $\sum_{i=1}^{N} \lambda_{i,t+1} = 1$ provide a system of N equations that pin down the date t+1 consumption shares $\{\lambda_{i,t+1}\}_{i=1,..,N}$ for given date t shares $\{\lambda_{i,t}\}_{i=1,..,N}$: $\lambda_{i+1,t+1} = \lambda_{i,t}(1-\lambda_{1,t})/(1-\lambda_{N,t})$ for i=1,..,N-1.

Assume that the consumption share of newborn agents, during the first period of their life, is time-invariant: $\lambda_{1,t} = \lambda_1 \ \forall t$. A constant newborn consumption share can be sustained by allocating to newborns a suitable time-invariant wealth share (see below). When $\lambda_{1,t} = \lambda_1$, then (A.1) is a stable difference equation in the consumption shares, and the consumption shares of generations i=2,..,N converge asymptotically to a constant consumption shares λ_i (numerical experiments show that convergence to the steady state shares is fast). The N steady state consumption shares obey

$$\lambda_{i+1} = \lambda_i (1-\lambda_1)/(1-\lambda_N) \text{ for i=1,..,N-1.} \quad \text{(A.3)}$$

Given any newborn's consumption share $0 < \lambda_1 \le 1$, these equations pin down unique consumption shares of generations i=2,..,N that are consistent with the adding up constraint

$\sum_{i=1}^{N}\lambda_i=1$. The following discussion assumes that the consumption shares equal their steady state values, so that all generational consumption shares are time-invariant: $\lambda_{1,t}=\lambda_1$ $\forall t,\ \forall i=1,..,N.$.

The Euler equation for capital of generation i=1,..,N-1 between periods t and t+1 is $E_t\rho_{t,t+1}r_{K,t+1}=1$, where $r_{K,t+1}$ is the gross rate of return (between t and t+1) on capital investment, while $\rho_{t,t+1}=\beta c_{i,t}/c_{i,t+1}$ is the common intertemporal marginal rate of substitution (IMRS) of these generations. Full risk sharing implies that the IMRS is equated across generations i=1,..,N-1 (see (A.1)). Thus

$$\rho_{t,t+1}=\beta\sum_{i=1}^{N-1}c_{i,t}/\sum_{i=1}^{N-1}c_{i,t+1}\quad\text{and}\tag{A.4}$$

$$\rho_{t,t+1}=\beta(C_t-c_{N,t})/(C_{t+1}-c_{1,t})=\beta[(1-\lambda_N)/(1-\lambda_1)]\cdot C_t/C_{t+1}.\tag{A.5}$$

The capital Euler equation can thus be expressed as

$$E_t\tilde{\beta}C_t/C_{t+1}r_{K,t+1}=1,\ \text{with}\ \tilde{\beta}\equiv\beta\times(1-\lambda_N)/(1-\lambda_1).\tag{A.6}$$

We thus see that, up to a rescaling of the subjective discount factor when $\lambda_1\neq\lambda_N$, this OLG model implies that the same 'aggregate' Euler equation (in terms of aggregate consumption) holds as in a model with an infinitely-lived representative agent. If the initial wealth endowment of newborns is such that $\lambda_1=1/N$, then $\lambda_i=1/N$ holds for i=1,..,N, which implies $\tilde{\beta}\equiv\beta$. In the special case where $\lambda_1=1/N$, the aggregate Euler equation of the OLG economy is thus *identical* to the Euler equation of an economy with an infinitely-lived agent. The only difference between the two economies is that the transversality condition $\lim_{\tau\to\infty}\beta^\tau E_t u'(C_{t+\tau})K_{t+\tau+1}=0$ does not hold in the OLG economy, as there is no infinitely-lived agent in the OLG economy. This OLG structure thus provides a motivation for considering a business cycle models that lack a TVC, but whose other equilibrium conditions (aggregate resource constraint, aggregate Euler equation) are identical to those of a standard business cycle model with an infinitely-lived representative agent.

Wealth shares

A time-invariant consumption share λ_1 of the new-born cohort is sustained by allocating to newborn agents a time-invariant share of the aggregate wealth of all cohorts. To see this, let $\omega_{i,t}$ denote the wealth of generation *i* in period t. $\omega_{i,t}$ equals the present value of generation *i*'s consumption stream: $\omega_{i,t}=E_t\sum_{s=0}^{N-i}\rho_{t,t+s}c_{i+s,t+s}$, where the stochastic discount factor $\rho_{t,t+s}$ is a product of the one-period-ahead discount factors defined in (A.5): $\rho_{t,t}=1$ and $\rho_{t,t+s}=\Pi_{\tau=1}^{\tau=s-1}\rho_{\tau,\tau+1}$ for $s>1$. Note that $\rho_{t,t+s}=\beta^s c_{i,t}/c_{i+s,t+s}$ for $0<s\leq N-i$. Therefore $\omega_{i,t}=c_{i,t}\sum_{s=0}^{N-i}\beta^s$ and hence

$$c_{i,t}=\phi_i\cdot\omega_{i,t},\ \text{with}\quad \phi_i\equiv(1-\beta)/(1-\beta^{N-i+1})\ \text{for i=1,..,N.}\tag{A.7}$$

Thus, in each period, generation i consumes a fraction ϕ_i of her wealth that is generation-specific, but time invariant. In an equilibrium with time-invariant generational consumption shares, the period t wealth of generation i equals thus $\omega_{i,t}=(\lambda_i/\phi_i)C_t$, and the wealth share generation i is

$$\omega_{i,t}/\sum_{s=1}^{N}\omega_{s,t}=(\lambda_i/\phi_i)/\sum_{s=1}^{N}(\lambda_s/\phi_s)\equiv\kappa_i\,. \tag{A.8}$$

Note that this wealth share is time-invariant. Thus, an equilibrium with time-invariant generational consumption shares exhibits time-invariant generational wealth shares. As pointed out above, the consumption share of newborn generations, λ_1, pins down uniquely the consumption shares of older generations, i.e. λ_i is a function of λ_1: $\lambda_i=\Lambda_i(\lambda_1)$. There is, hence, a unique mapping from λ_1 to the wealth shares of all generations (see (A.8) for definition of κ_i):

$$\kappa_i=\mathrm{K}_i(\lambda_1)=(\Lambda_i(\lambda_1)/\phi_i)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)\,. \tag{A.9}$$

If the new-born generation is allocated a wealth share $\kappa_1=(\lambda_1/\phi_1)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)$, then this sustains an equilibrium in which the consumption share of the new-born generation is λ_1. A consumption allocation in which all generations have consumption share $\lambda_i=1/N$ is sustained by allocating to the newborn generation a wealth share $\kappa_1=(1/\phi_1)/\sum_{s=1}^{N}1/\phi_s$. As an example, assume that life lasts 80 years, i.e. N=320 quarters, and that the quarterly subjective discount factor is β=0.99; then the consumption allocation with equal consumption shares $\lambda_i=1/N=0.3125\%$ requires a newborn wealth share of $\kappa_1=0.4267\%$.

Appendix 2: Bubble equilibrium in RBC model with incomplete capital depreciation (constant TFP)

This Appendix discusses the role of Δ for the bubble equilibrium, in the RBC model with constant TFP. Recall that Δ denotes the deviation of the capital stock selected in the bust state, from the no-bubbles decision rule, λ: $K_{t+1}^{L}=\lambda(K_t)e^{\Delta}$.

Consider first a decision economy with Δ=0, so that $K_{t+1}=\lambda(K_t)\ \ \forall t$. Then the agent's Euler equation holds between t and t+1: $H(\lambda(\lambda(K_t)),\lambda(K_t),K_t)=1$. If $K_{t+1}=\lambda(K_t)e^{\Delta}\ \ \forall t$ the Euler equation fails to hold if $\Delta\neq 0$. Specifically:

$$H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)<1 \text{ when } \Delta>0, \tag{A.10}$$

$$\text{while } H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)>1 \text{ when } \Delta<0. \tag{A.11}$$

(Intuitively, Δ>0 implies overinvestment in capital, and thus the intertemporal marginal rate of transformation is smaller than the intertemporal marginal rate of transformation, IMRS, which implies H<1; Δ<0 implies underinvestment in capital, and thus the intertemporal marginal rate of transformation is greater than the IMRS and hence H>1.)

I now discuss bubble equilibria. Recall that the bubble equilibria considered here are such that the capital stock set at date t takes two possible values: $K_{t+1}\in\{K_{t+1}^{L},K_{t+1}^{H}\}$ with exogenous probabilities π and $1-\pi$, respectively, where $K_{t+1}^{L}=\lambda(K_t)e^{\Delta}$. I now show that a bounded equilibrium with recurrent bubbles exists if $\Delta>0$. When $\Delta=0$, the bubble equilibrium is self-ending. When $\Delta<0$ the Euler equation between *t* and *t*+1 fails to have a solution for K_{t+2}^{H}, for certain values of K_t,K_{t+1}. Thus, there is not bubble equilibrium if Δ<0.

I) Consider first a situation in which the date t+1 capital stock equals K_{t+1}^{L}: $K_{t+1}=\lambda(K_t)e^{\Delta}$.
Then $K_{t+2}^{L}=\lambda(\lambda(K_t)e^{\Delta})e^{\Delta}$ and the Euler equation (17) between periods *t* and *t*+1 becomes:

$$\pi H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)+(1-\pi)\cdot H(K_{t+2}^{H},\lambda(K_t)e^{\Delta},K_t)=1. \tag{A.12}$$

To establish the existence of a 'bubble equilibrium', one needs to show that there exists a $K_{t+2}^{H}\in(K_{\Delta}^{\min},K^{\max})$ that solves (A.12).

● Consider first the case where Δ=0. Recall that $H(\lambda(\lambda(K_t)),\lambda(K_t),K_t)=1$. Thus, for Δ=0, the Euler equation of the bubbly economy (A.12) requires that $H(K_{t+2}^{H},\lambda(K_t),K_t)=1$ holds. This implies $K_{t+2}^{H}=\lambda(\lambda(K_t))$, and thus $K_{t+2}=K_{t+2}^{H}=K_{t+2}^{L}=\lambda(K_{t+1})$. By the same logic, $K_{s+1}=\lambda(K_s)$ has to hold $\forall s\geq t+1$. Thus, if $K_{s+1}=\lambda(K_s)$, then the agent has to continue sticking to the no-bubbles decision rule in all subsequent periods. Hence, the bubble is self-ending when Δ=0.

● Consider next the case Δ>0. Because $H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)<1$ when Δ>0, the Euler equation (A.12) can only holds when $H(K_{t+2}^{H},\lambda(K_t)e^{\Delta},K_t)>1$. Note that $H(K_{t+2}^{H},\lambda(K_t)e^{\Delta},K_t)<1$ when $K_{t+2}^{H}=\lambda(\lambda(K_t)e^{\Delta})e^{\Delta}$. It can be verified that $H(K_{t+2}^{H},\lambda(K_t)e^{\Delta},K_t)$ is an increasing function of K^H_{t+2} (as a rise in K^H_{t+2} lowers C_{t+1} and raises hours worked L_{t+1} which raises the marginal utility of consumption at t+1, and raises the marginal product of capital at t+1). Setting K^H_{t+2} arbitrarily close to (but below) the maximum feasible value $\theta(\lambda(K_t)e^{\Delta})^{\alpha}+(1-\delta)\lambda(K_t)e^{\Delta} < K^{\max}$ makes C_{t+1} very close to zero (which implies that L_{t+1} is very close to 1), which makes $H(K^H_{t+2},\lambda(K_t)e^{\Delta},K_t)$ very big. This implies that there exists a unique value of K^H_{t+2} that solves the Euler equation (A.12). Note that $K^H_{t+2}\in(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},K^{\max})$. Thus, $K^L_{t+2}<K^H_{t+2}$. When $K_t\in(K^{\min}_{\Delta},K^{\max})$, then $\lambda(K_t)e^{\Delta}\in(K^{\min}_{\Delta},K^{\max})$ and $\lambda(\lambda(K_t)e^{\Delta})e^{\Delta}\in(K^{\min}_{\Delta},K^{\max})$ for values of $\Delta>0$ sufficiently close to 0. If $K_t\in(K^{\min}_{\Delta},K^{\max})$ we thus have that $K^L_{t+1},K^L_{t+2},K^H_{t+2}\in(K^{\min}_{\Delta},K^{\max})$.

● Finally, consider the case $\Delta<0$. It follows from (A.11) that then the Euler equation (A.12) requires that $H(K^H_{t+2},\lambda(K_t)e^{\Delta},K_t)<1$ holds. If there exists a value of K^H_{t+2} that solves (A.12), then that value must be smaller than K^L_{t+2}: $K^H_{t+2}<K^L_{t+2}=\lambda(\lambda(K_t)e^{\Delta})e^{\Delta}$ when $\Delta<0$. There is no solution for K^L_{t+2} when $\pi H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)+(1-\pi)\cdot H(0,\lambda(K_t)e^{\Delta},K_t)>1$. When $\Delta<0$, then a succession of positive (!) draws of the sunspot (u>0) puts the capital stock on a downward trajectory until (A.12) cannot be solved anymore for $K^H_{t+2}\geq 0$.

II) Consider next a situation in which the date t+1 capital stock equals K^H_{t+1}: $K_{t+1}=K^H_{t+1}$

Assume $\Delta>0$. As shown above, $K^H_{t+1}\geq\lambda(K_t)e^{\Delta}$ holds when $\Delta>0$. When $K_{t+1}=K^H_{t+1}$ (which is triggered by a positive realization of the date *t* sunspot, $u_t>0$), then $K^L_{t+2}=\lambda(K^H_{t+1})e^{\Delta}$ and the Euler equation between periods *t* and *t*+1 is given by:

$$\pi H(\lambda(K^H_{t+1})e^{\Delta},K^H_{t+1},K_t)+(1-\pi)\cdot H(K^H_{t+2},K^H_{t+1},K_t)=1. \tag{A.13}$$

$H(\lambda(K^H_{t+1})e^{\Delta},K^H_{t+1},K_t)$ is a decreasing function of K^H_{t+1} for $K^H_{t+1}\geq\lambda(K_t)e^{\Delta}$. Recall that $H(\lambda(\lambda(K_t)e^{\Delta})e^{\Delta},\lambda(K_t)e^{\Delta},K_t)<1$ when $\Delta>0$. Therefore $H(\lambda(K^H_{t+1})e^{\Delta},K^H_{t+1},K_t)<1$ for any $K^H_{t+1}\geq\lambda(K_t)e^{\Delta}$. Thus $H(K^H_{t+2},K^H_{t+1},K_t)>1$. It follows from the discussion above that $H(K^H_{t+2},K^H_{t+1},K_t)$ is increasing in K^H_{t+2} and that $H(K^H_{t+2},K^H_{t+1},K_t)$ can be made arbitrarily big by setting K^H_{t+2} close to $\theta(K^H_{t+1})^{\alpha}+(1-\delta)K^H_{t+1}$. Thus, there exists a unique K^H_{t+2} that solves (A.13). Furthermore, $K^H_{t+2}>K^L_{t+2}\equiv\lambda(K^H_{t+1})e^{\Delta}$.

Table 1. Long-Plosser model with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Specification I: $Z_t^L = \alpha\beta + \Delta$												
π_t=0.5	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t \cong 1$ for z_t>0.36	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
(b) US Data (from King and Rebelo (1999))												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				

Note: all business statistics pertain to HP-filtered logged variables.

Table 2. RBC model (incomplete capital depreciation): predicted business cycle statistics

	Bubbles, no TFP shocks				Bubbles & TFP shocks				Just TFP shocks		
	Unit Risk aversion		High RA		Unit RA		High RA				
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.27	1.60	0.98	1.57	1.14	0.72	1.81
C	1.08	2.63	0.29	0.61	1.16	2.71	0.38	0.72	0.49	0.26	1.35
I	4.29	9.38	3.22	6.51	5.38	9.85	3.86	6.72	3.33	2.20	5.30
L	0.74	1.73	1.04	2.18	0.82	1.70	1.05	2.22	0.34	0.30	1.79
Correlations with GDP											
C	-0.97	-0.95	-0.99	-0.98	0.04	-0.54	0.01	-0.62	0.95	0.99	0.88
I	0.98	0.96	0.99	0.99	0.89	0.86	0.97	0.98	0.99	0.99	0.80
L	0.99	0.97	0.99	0.99	0.79	0.81	0.45	0.82	0.98	-0.96	0.88
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.65	0.68	0.57	0.66	0.71	0.70	0.84
C	0.33	0.60	0.35	0.62	0.43	0.62	0.53	0.65	0.76	0.72	0.80
I	0.36	0.63	0.37	0.64	0.53	0.65	0.51	0.65	0.70	0.70	0.87
L	0.34	0.61	0.35	0.62	0.45	0.62	0.41	0.63	0.70	0.74	0.88
Means [% deviation from steady state]											
Y	1.41	2.80	1.25	2.12	1.37	2.75	1.31	2.17	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.68	1.34	0.33	0.55	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.61	7.28	4.44	7.40	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.34	0.73	0.01	-0.03	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.16	8.78	8.92	8.53	9.58	9.58	13.42
Fraction of periods with (capital income > investment) [in %]											
	99.20	96.31	99.55	97.72	99.20	96.43	99.37	97.74	100	100	100

Notes: Business cycle statistics reported here are based on one simulation run of T=10000 periods (for each model variant). Standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. These moments pertain to logged series that were HP filtered (for each window of 200 periods). "Means" are sample averages over the total sample of T periods. The "Fraction of periods with (capital income > investment)" likewise pertains to the whole simulation run of T periods. Cols. (1)-(4) pertain to model variants in which fluctuations are just driven by bubbles (constant TFP). Cols. (5)-(8) pertain to variants with bubbles and TFP shocks. Cols. (9)-(10) assume just TFP shocks (without bubbles). Col. (11) reports empirical statistics based on US data (statistics for Y,C,I,L: from King and Rebelo (1999), based on 1947q1-1996q4 quarterly data; statistics about capital income – investment: based on annual data 1929-1985 reported by Abel et al. (1989)).

Table 3. Two-country Dellas model: predicted business cycle statistics

	Bubble; no TFP shocks	Bubble & TFP shocks	Just TFP shocks	Data
	(1)	*(2)*	*(3)*	*(4)*
Standard deviations [in %]				
Y	22.08	22.11	1.38	1.81
C	77.93	77.81	1.29	1.35
I	46.75	46.73	1.29	5.30
L	15.98	15.98	0.00	1.79
RER	0.00	1.21	1.23	3.66
NX	0.00	0.00	0.00	0.46
Correlations with domestic GDP				
C	-0.65	-0.65	0.99	0.88
I	0.89	0.89	0.99	0.80
L	0.76	0.76	--	0.88
RER	--	-0.03	-0.57	-0.23
NX	--	--	---	-0.56
Autocorrelations				
Y	0.72	0.72	0.80	0.84
C	0.34	0.34	0.81	0.80
I	0.59	0.59	0.81	0.87
L	0.43	0.43	--	0.88
RER	--	0.75	0.75	0.80
NX	--	--	--	0.78
Cross-country correlations				
Y	1.00	0.99	0.37	0.52
C	1.00	0.99	0.55	0.35
I	1.00	0.99	0.55	0.38
L	1.00	1.00	--	0.43
Means [% deviation from steady state]				
Y	29.16	29.05	-0.01	--
C	-0.02	-0.02	-0.01	--
I	92.14	91.86	-0.01	--
L	12.51	12.51	0.00	--
Mean (capital income – investment)/GDP [in %]				
	-10.67	-10.67	0.33	13.42
Fraction of periods with (capital income > investment) [in %]				
	0.00	0.00	100.00	100.00

Notes: This Table reports simulated business cycle statistics for a two-country RBC model with full capital depreciation (see Sect. 4 of paper). Col. (1) pertains to a model variant in which fluctuations are just driven by bubbles (constant TFP). Col. (2) considers a model variant with simultaneous bubbles and TFP shocks. Col. (3) assumes just TFP shocks (without bubbles). The bubble process assumes a bust

probability π=0.5. Simulated business cycle statistics are based on one simulation run of T=10000 periods (for each model variant). Simulated standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. These moments pertain to series that were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). "Means" are sample averages over the total sample of T periods. The "Fraction of periods with (capital income > investment)" likewise pertains to the whole simulation run of T periods.
Col. (11) reports empirical statistics. Historical standard deviations, correlations with domestic GDP and autocorrelations of GDP, consumption, investment and hours worked, as well as the statistics on capital income-investment, correspond to statistics based on (logged and HP filtered) US data reported in Table 2 (see sources indicate there). The 'international' empirical statistics are based on quarterly data for 1973q1-2013q4: the reported moments of RER and NX (net exports/GDP) pertain to US data (RER: from BIS; NX: from BEA); Cross-country correlations of Y,C,I,L are correlations between US and Euro Area data, 1973q1-2013q4 (EA data: from Euro Area Wide Model database).

Table 4. International RBC model (incomplete capital deprec.): predicted business cycle statistics

	Bubbles, no TFP shocks				Bubbles & TFP shocks						
	Unit Risk aversion		High RA		Unit RA		High RA		Just TFP shocks		
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.46	1.78	1.18	1.65	1.32	0.97	1.81
C	1.08	2.63	0.29	0.61	1.18	2.79	0.41	0.70	0.56	0.31	1.35
I	4.29	9.38	3.22	6.51	6.36	10.54	4.95	7.34	4.60	3.90	5.30
L	0.74	1.73	1.04	2.18	0.88	1.79	1.13	2.24	0.44	0.62	1.79
RER	0.00	0.00	0.00	0.00	0.32	0.32	0.44	0.44	0.32	0.44	3.66
NX	0.00	0.00	0.00	0.00	0.16	0.16	0.14	0.14	0.16	0.13	0.46
Correlations with domestic GDP											
C	-0.97	-0.95	-0.99	-0.98	0.09	-0.46	0.03	-0.55	0.85	0.61	0.88
I	0.98	0.96	0.99	0.99	0.90	0.88	0.97	0.98	0.95	0.96	0.80
L	0.99	0.97	0.99	0.99	0.81	0.81	0.46	0.78	0.94	-0.01	0.88
RER	--	--	--	--	-0.44	-0.35	-0.58	-0.39	-0.48	-0.68	-0.23
NX	--	--	--	--	-0.53	-0.46	-0.58	-0.46	-0.58	-0.68	-0.56
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.63	0.67	0.57	0.65	0.67	0.64	0.84
C	0.33	0.60	0.35	0.62	0.46	0.62	0.57	0.65	0.75	0.71	0.80
I	0.38	0.63	0.37	0.64	0.54	0.64	0.55	0.64	0.63	0.61	0.87
L	0.34	0.61	0.35	0.62	0.46	0.62	0.48	0.64	0.63	0.69	0.88
RER	--	--	--	--	0.84	0.82	0.80	0.79	0.84	0.81	0.80
NX	--	--	--	--	0.61	0.62	0.65	0.66	0.61	0.66	0.78
Cross-country correlations											
Y	1.00	1.00	1.00	1.00	0.29	0.54	-0.00	0.52	0.17	-0.46	0.52
C	1.00	1.00	1.00	1.00	0.96	0.99	0.98	0.99	0.84	0.96	0.35
I	1.00	1.00	1.00	1.00	0.27	0.74	-0.07	0.53	-0.35	-0.83	0.38
L	1.00	1.00	1.00	1.00	0.63	0.92	0.85	0.96	-0.35	0.46	0.43
Means [% deviation from steady state]											
Y	1.41	2.80	1.25	2.12	1.65	3.02	1.45	2.29	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.95	1.60	0.44	0.65	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.93	7.61	4.72	7.61	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.35	0.73	0.09	0.05	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.15	8.78	8.89	8.51	9.55	9.58	13.42
Fraction of periods with (capital income > investment) [in %]											
	99.20	96.31	99.55	97.72	99.20	96.45	99.44	97.75	100	100	100

Notes: This Table reports simulated business cycle statistics for different variants of the two-country RBC model with incomplete capital depreciation (see Sect. 5 of paper). Cols. (5)-(8) pertain to model variants with bubbles and TFP shocks. Cols. (9)-(10) assume just TFP shocks (without bubbles). In Cols. (1)-(8), π refers to the bust probability. 'Unit

Risk aversion' (RA): model versions with log utility. 'High RA': model version in which consumption utility is $\ln(C_{i,t} - \overline{C})$, where $\overline{C}>0$ is a constant. Business cycle statistics are based on one simulation run of T=10000 periods, for each model variant. Standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. These moments pertain to series that were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). "Means" are sample averages over the total sample of T periods. The "Fraction of periods with (capital income > investment)" likewise pertains to the whole simulation run of T periods. Cols. (1)-(4) pertain to model variants in which fluctuations are just driven by bubbles (constant TFP).
Col. (11) reports empirical statistics. See Table 3 for information and data sources.

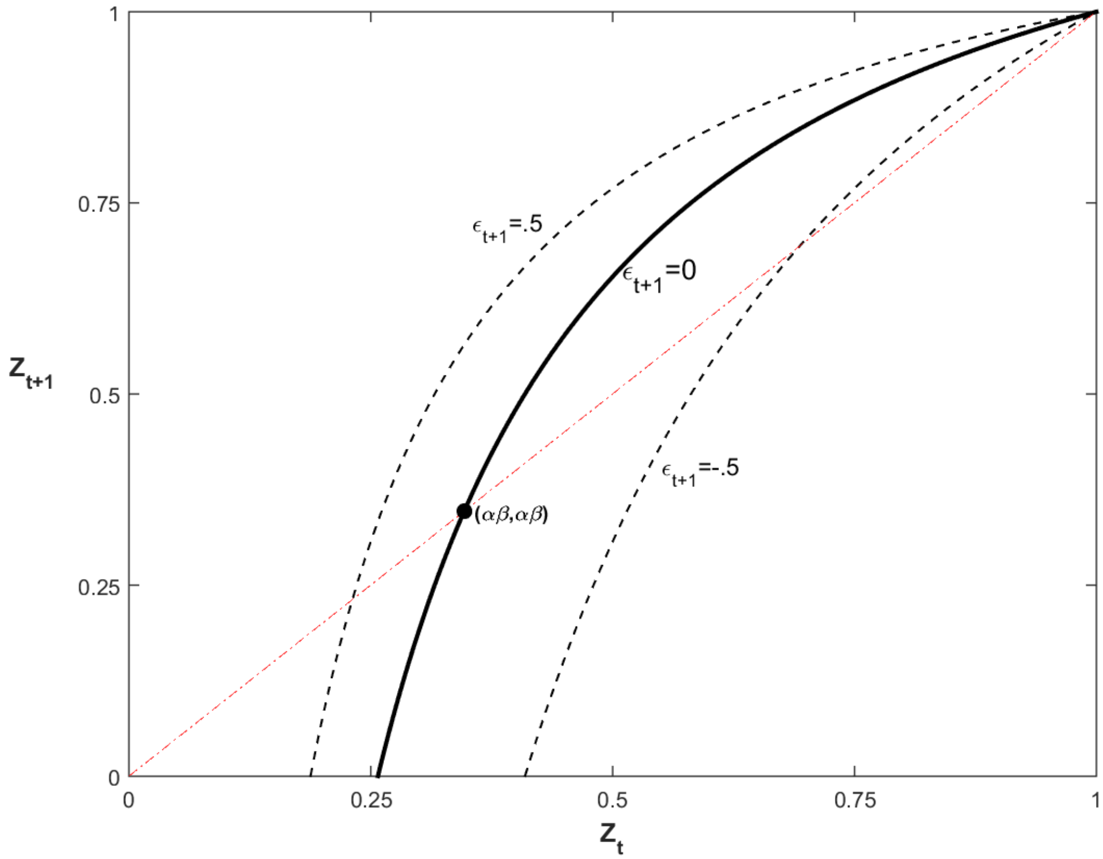
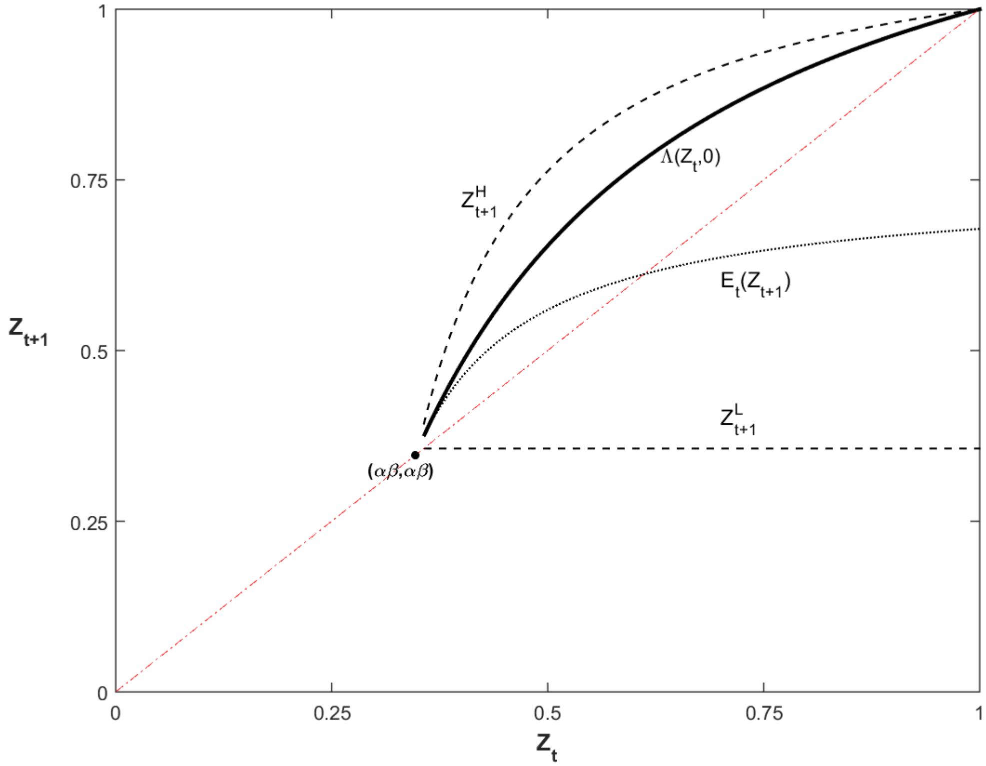
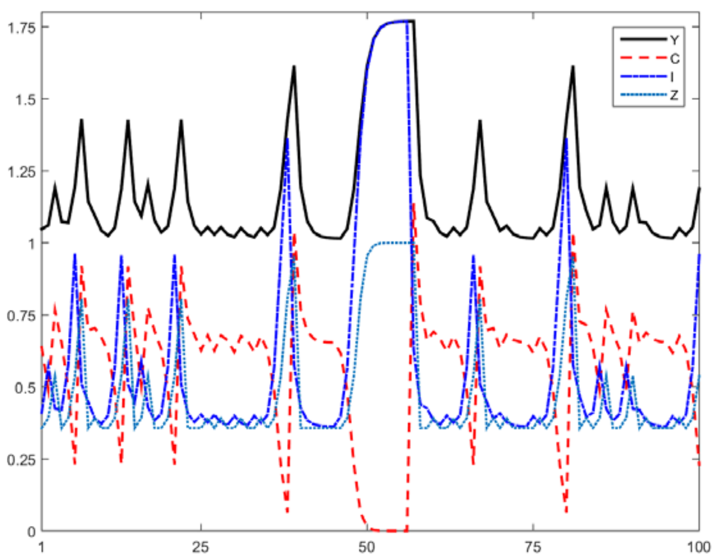


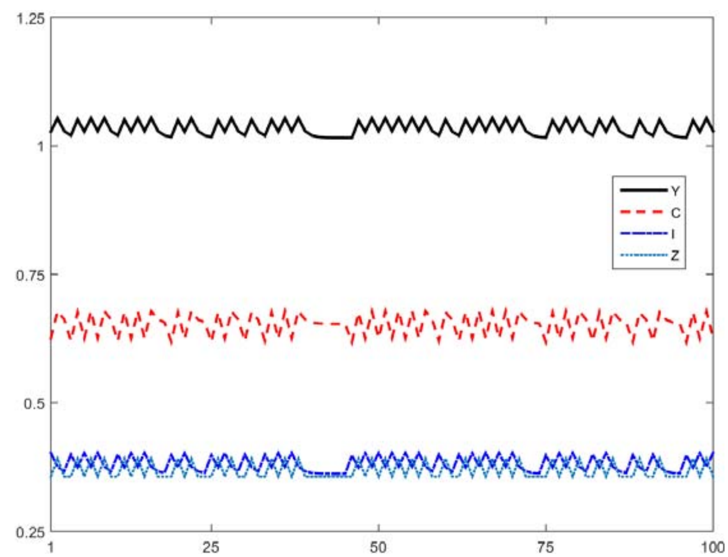
Figure 1. Long & Plosser model: investment/output ratio at t+1, Z_{t+1}, as a function of Z_t for $\varepsilon_{t+1} \in \{-0.5; 0; 0.5\}$



(a) 'Low' and 'High' values of date t+1 investment/output ratio (Z_{t+1}^L, Z_{t+1}^H) and expected value $(E_t Z_{t+1})$ shown as function of $Z_t \in [\alpha\beta+\Delta, 1)$. $\Lambda(Z_t,0)$ is value of Z_{t+1} without random sunspot. Probability of 'Low' value Z_{t+1}^L: $\pi_t=0.5$ $\forall t$ $Z_t \in [\alpha\beta+\Delta, 1)$



(b) Simulated series with constant probability: $\pi_t=0.5$.



(c) Simulated series with $\pi_t=0.5$ for $Z_t \leq 0.36$ and $\pi_t \simeq 1$ for $Z_t > 0.36$

Figure 2. Long & Plosser model with bubbles.
Simulated series of output (Y), consumption (C) and investment are normalized by steady state output.

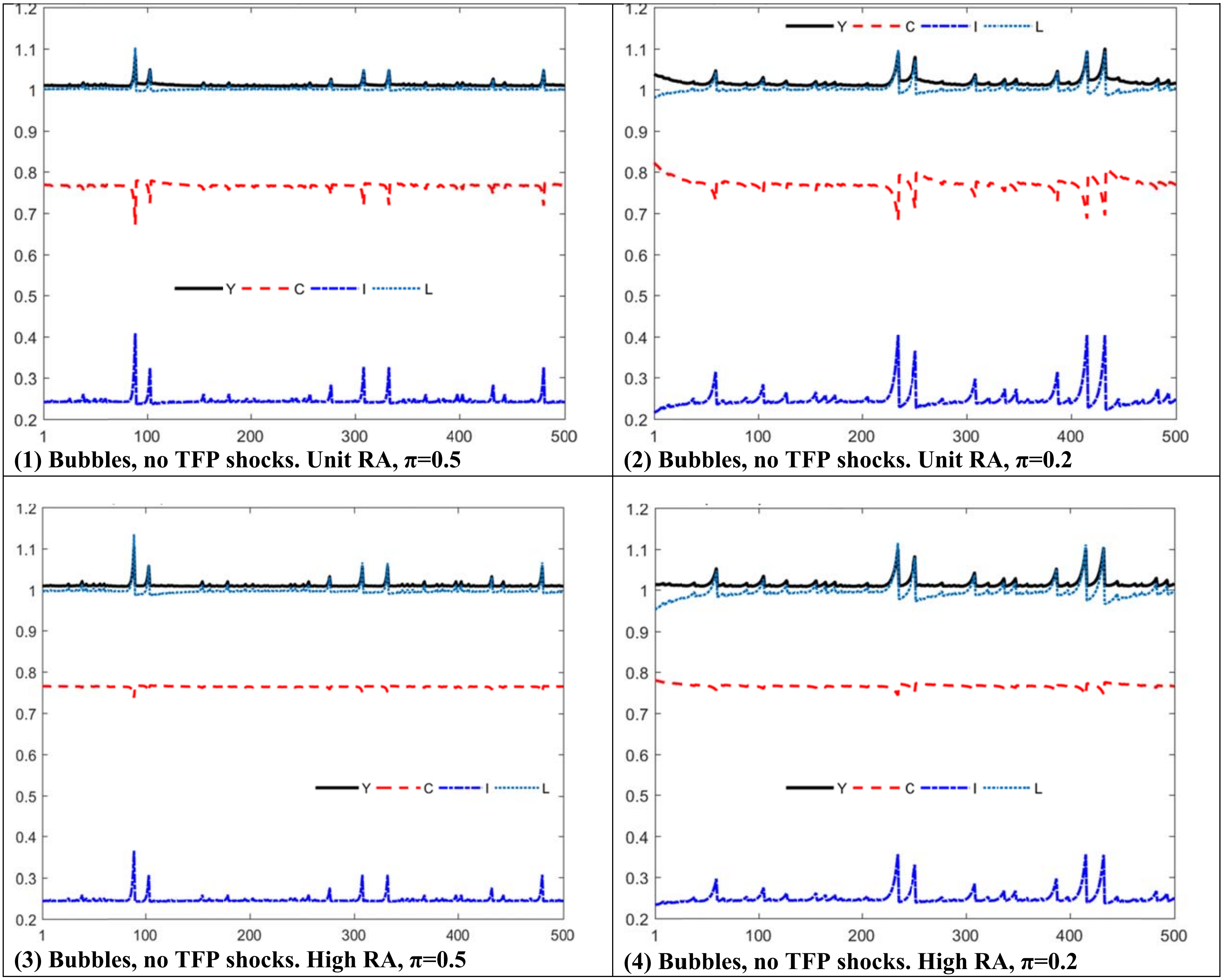
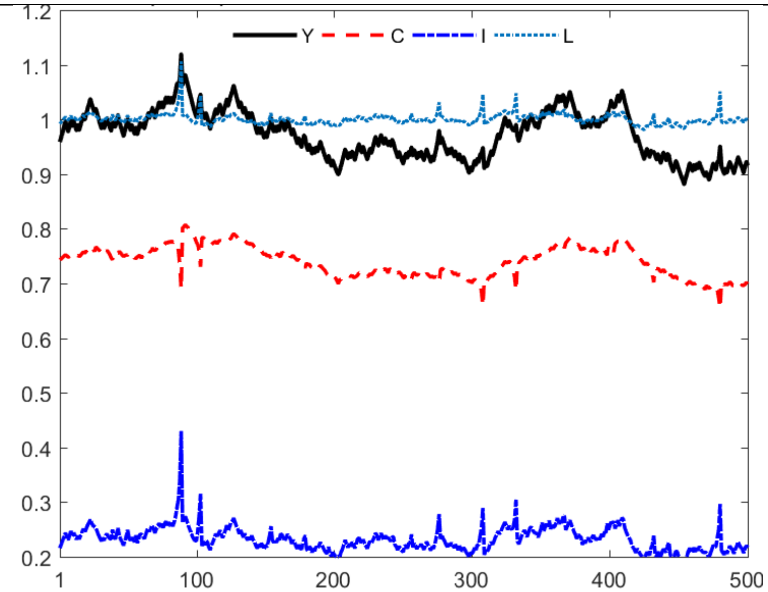
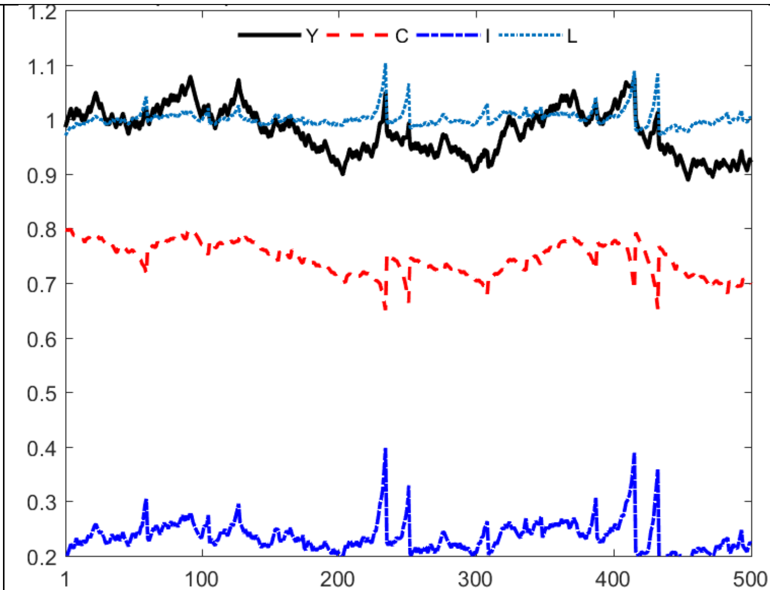


Figure 3. Non-linear RBC model (incomplete capital depreciation): simulated paths

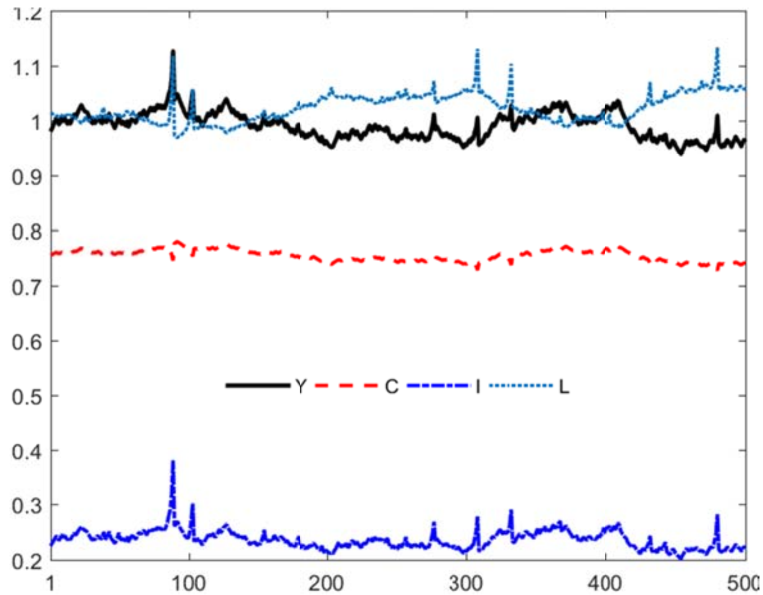
Simulated paths of GDP (Y, continuous black line), consumption (C, red dashed line), investment (I, blue dash-dotted line) and hours worked (L, blue dotted line) are shown for 10 variants of the RBC model with incomplete capital depreciation and variable labor described in Sect. 3. Panel (i) of this Figure assumes the model variant considered in Col. (i) of Table 2. RA: risk aversion. Panels (1)-(4) assume that bubbles are the only driving force (TFP is constant); Panels (5)-(8) assume joint bubble and TFP shocks; Panel (9)-(10) assume just TFP shocks. GDP, C and I series are normalized by steady state GDP. The hours worked series is normalized by steady state hours.



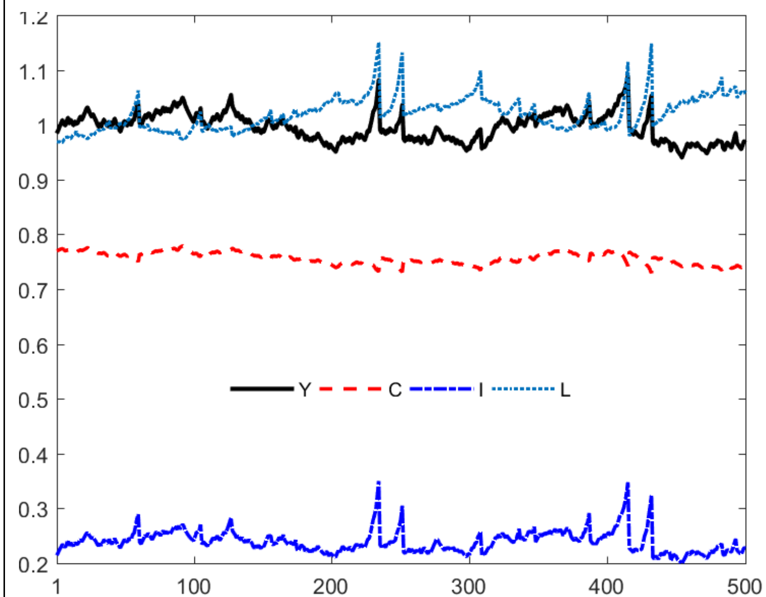
(5) Bubbles & TFP shocks. Unit RA, π=0.5



(6) Bubbles & TFP shocks. Unit RA, π=0.2

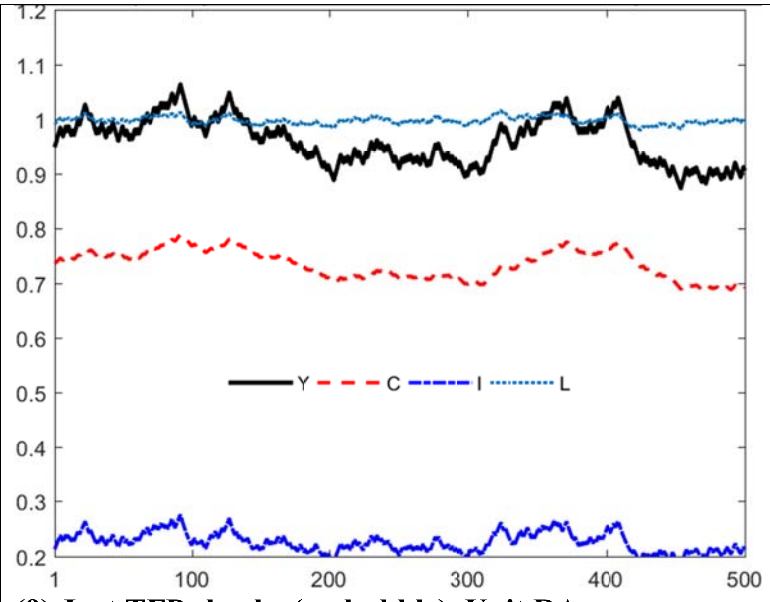


(7) Bubbles & TFP shocks. High RA, π=0.5

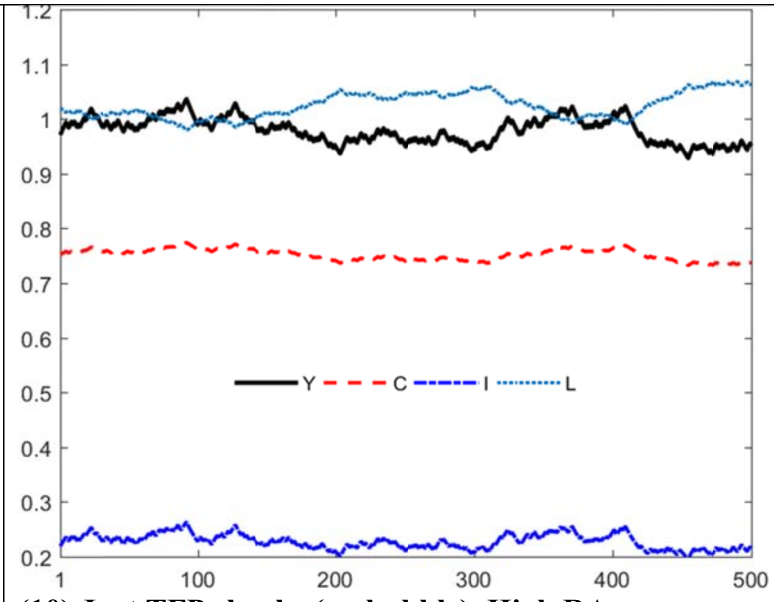


(8) Bubbles & TFP shocks. High RA, π=0.2

Figure 3. (continued)



(9) Just TFP shocks (no bubble). Unit RA



(10) Just TFP shocks (no bubble). High RA

Figure 3. (continued)

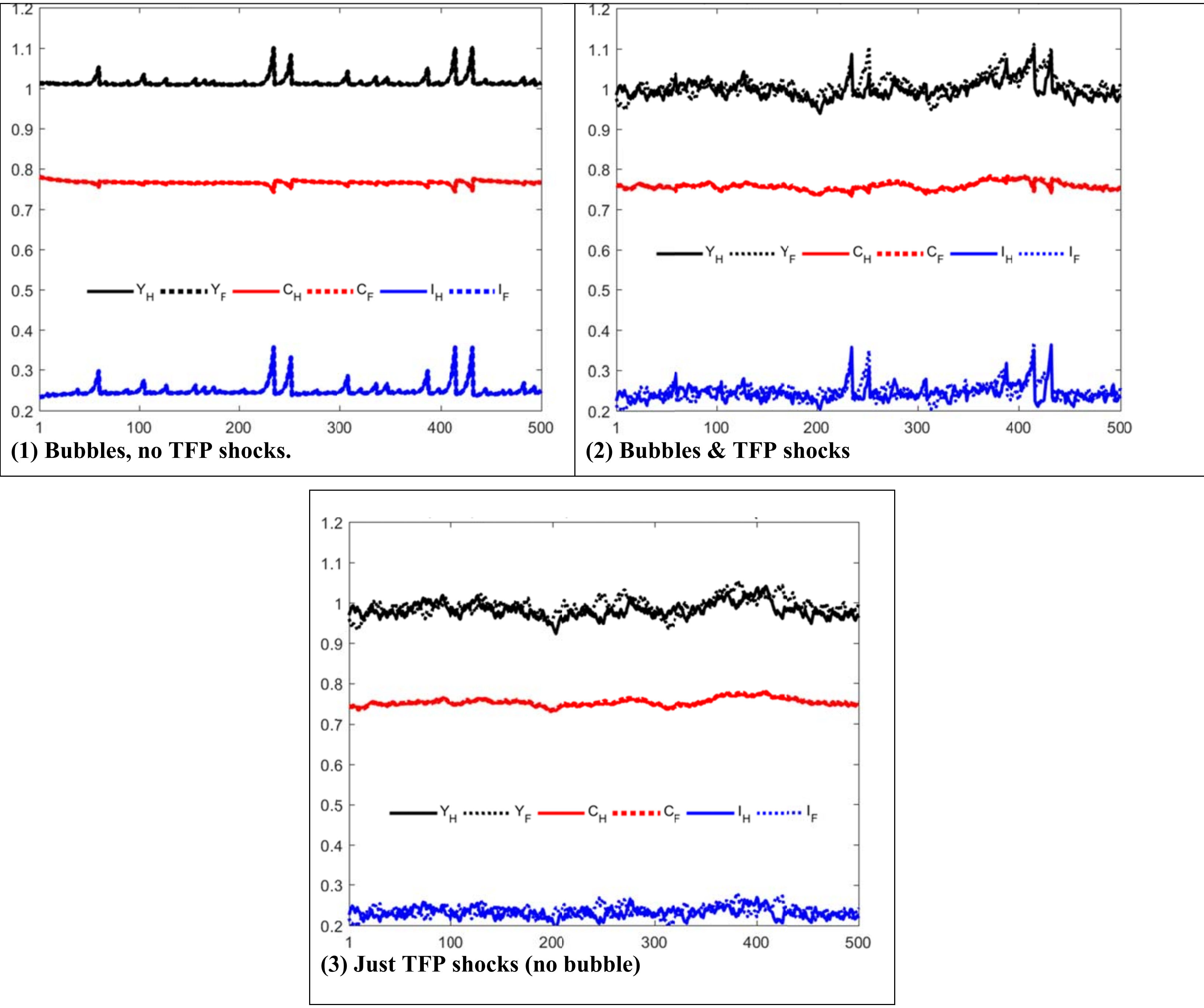


Figure 4. Non-linear International RBC model (incomplete capital depreciation): simulated paths

The Figure assumes the two-country RBC model with incomplete capital depreciation and 'High risk aversion' and a bust probability π=0.20. Simulated paths of Home and Foreign GDP (Y_H, Y_F: continuous and dotted black line), Home and Foreign consumption (C_H, C_F: continuous and dotted red line) and investment (I_H, I_F: continuous and dotted blue lines). Panel (1) assumes that bubbles are the only driving force (TFP is constant); Panel (2) assume joint bubble and TFP shocks; Panel (3) assume just TFP shocks (no-bubble solution). GDP, C and I series are normalized by steady state GDP.